Archimedes' Law of the Lever, and his Mysterious Mechanical Method for Finding the Volume of a Sphere

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Attention iPad Users!

The iPad renders three slides incorrectly —

The ones with captions beginning:

“Cylinder”, “Cone”, and “Sphere”.

(Verified Mar 24, 2012)

Archimedes:

He invented physical modeling

and the mathematics needed to do it!

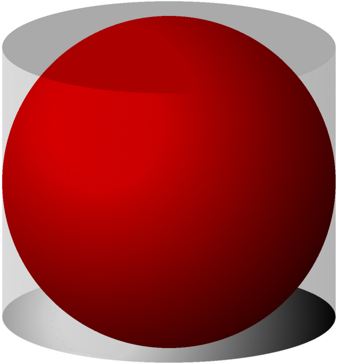
Little is known about him.

- Archimedes of Syracuse, 287 ? – 212 BC
- Greek mathematician, physicist, engineer, inventor and astronomer
- Approximated π, determined the area of a circle and the volume of a sphere in terms of π
- Invented the compound pulley and *explained* the mechanical advantage of the lever
- Laid foundations in hydrostatics and statics, calculated area of parabola using summation of an infinite series, and defined the spiral of Archimedes
- Killed by a Roman soldier during the capture of Syracuse.

Archimedes and the Roman Soldier

(Anon.)

Archimedes' Proudest Achievement



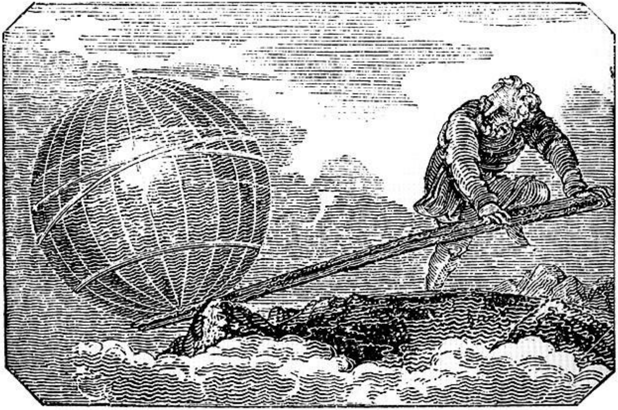
(NYU, http://math.nyu.edu/ crorres/Archimedes/Tomb/Cicero.html)

The enclosed sphere has $2/3$ the volume of the cylinder.
In this talk we begin with the Law of the Lever, then conclude with Archimedes' use of it to determine the volume of a sphere.

Part 1

The Law of the Lever

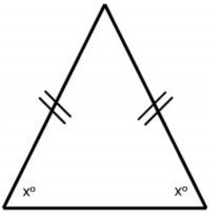
Archimedes' Moves the World



(Anon.)

"Give me a lever long enough and a place to stand and I will move the world."

Archimedes' Law was innovative like the "Pons Asinorum" of Thales.

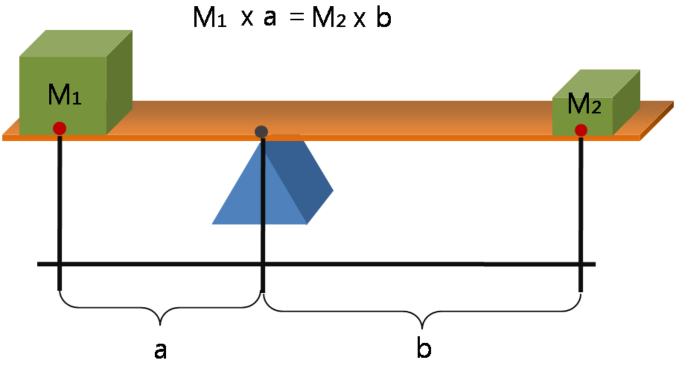


Theorem: Base angles of an isosceles triangle are equal. Seems obvious.

But Greeks wanted strictly deductive proofs based on stated axioms — not loose arguments like donkeys prefer.

Pons Asinorum, the Bridge of Asses. Theorem attributed to Thales (c 624 – 526 BC).

The Law of the Lever

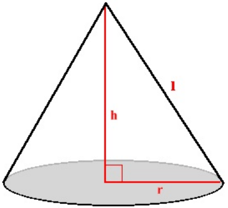


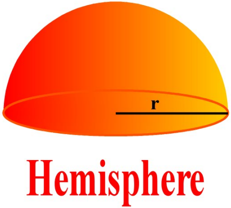
(Wikipedia)

Archimedes assumed: A mass presses down on a static beam as if concentrated at its *center-of-mass*.

The beam is stiff and weightless.

Archimedes invented the Center-of-Mass.





(Clip Art Freeware)

Where is the center-of-mass? This usually needs calculus, but Archimedes understood the concept without *modern* calculus.

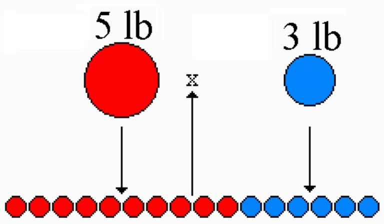
The Axiom of Equivalence

A distributed mass balances like the same mass concentrated at its center-of-mass.

This is an empirical fact not deducible from geometry.

The Law of the Lever rests on the concept of center-of-mass.

Archimedes' Proof of the Law of the Lever



(http://physics.weber.edu/carroll/archimedes/leverprf.htm)

To find a balance point between two weights, divide them into commensurable units, colored red and blue. Place the units uniformly on a beam; the balance point is at the center of the units. The proportionate distances are measured from the center of the units, 3 : 5 in this case. So, 5 x 3 = 3 x 5.

Archimedes' Innovations

Archimedes:

1. invented the concept of center-of-mass;
2. introduced a rigorous mathematical model to describe a physical phenomenon: the Law of the Lever;
3. applied his law of the lever to find the volume of a sphere — this was the mysterious Mechanical Method he referred to in his correspondence with other geometers

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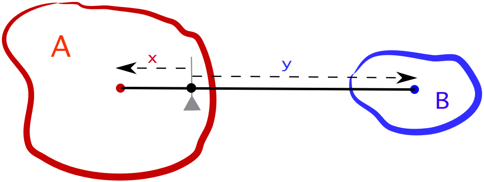
Axioms for Center-of-Mass for Areas



Archimedes generalized the Law of the Lever. Figure illustrates two axioms for balancing *areas* in 2D:

$$(\text{Area } A) \cdot x = (\text{Area } B) \cdot y$$

Third axiom: center-of-mass of a convex area lies within the area.

Archimedes used these in his theory of bouyancy.

See Stein in *Recommended reading*.

The Modern Definition of Center-of-Mass

Given point-masses $\{(m_i, \mathbf{x}_i), i = 1, \ldots, n\}$, their center of mass $\overline{\mathbf{X}}$, also called centroid, is defined

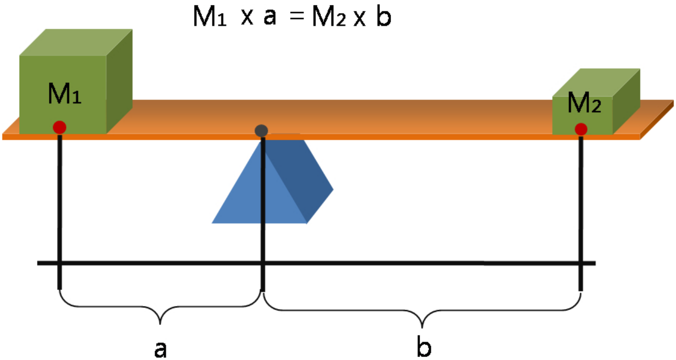
$$\overline{\mathbf{X}} = \frac{\sum_{i=1}^{n} m_i \mathbf{x}_i}{\sum_{i=1}^{n} m_i}$$

More generally,

$$\overline{\mathbf{X}} = \frac{\int_V \mathbf{x} dm}{\int_V dm}$$

These vector equations, which yield the balance point for a mass distribution in 1-D, generalize the Law of the Lever. In 3-D they are fundamental constructs of rigid-body mechanics.

The Law of the Lever



(Wikipedia)

Part II

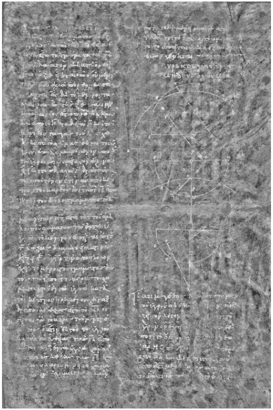
Volume of the Sphere

Archimedes' Method, unknown until 1906, applies the Law of the Lever.

Archimedes explained his "Mechanical Method" in a Palimpsest discovered in 1906 in a Byzantine Crypt in Istanbul.

Here's what it looked like when rediscovered in 1998.

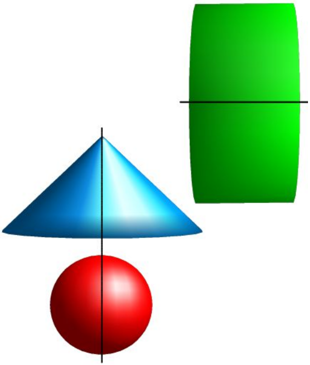
One Imaged Page of the Archimedes Palimpsest



(Wikipedia)

In the early 1900s, the Danish philologist Johan Heiberg transcribed legible portions into Greek. Here is a recent scan.

Archimedes used the Law of the Lever to compare volumes of a cylinder, cone and sphere.

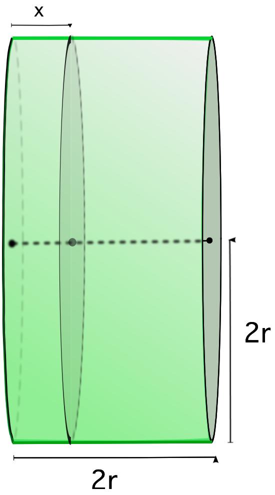


Archimedes knew the volumes of cylinders and cones. and areas of their circular cross sections. He used these to determine the volume of the sphere.

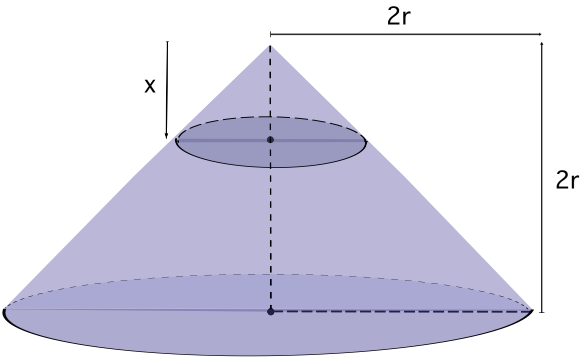
Cylinder: Area of x-level Cross Section

Circular cross-section at level x: $4\pi r^2$

[Volume of cylinder = $C_y = 8\pi r^3$]



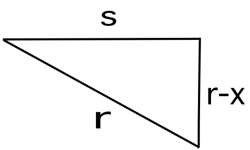
Cone: Area of x-level Cross Section

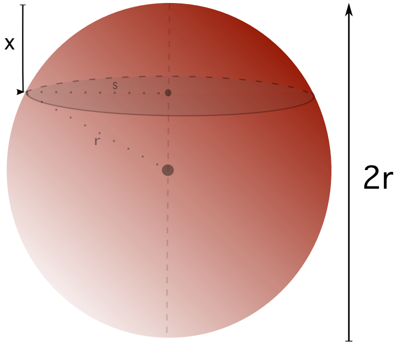


Circular cross-section at level x: πx^2

[Volume of cone = $C_o = \frac{8}{3}\pi r^3$]

Sphere: Area of x-level Cross Section

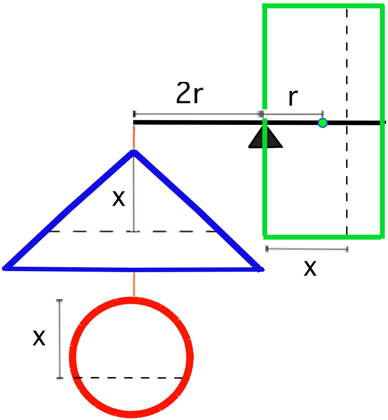




Circular cross-section at level x: $\pi s^2 = 2\pi rx - \pi x^2$

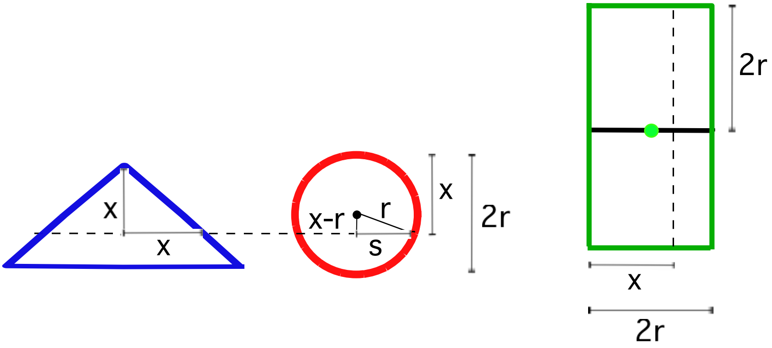
[Volume of sphere = $S =$???]

Method (Step 1): Three Solids in Perfect Balance



We will see that the *x*-level cross sections balance. (Assuming uniform density.)

Method (Step 2): The cross-section (CS) areas balance.



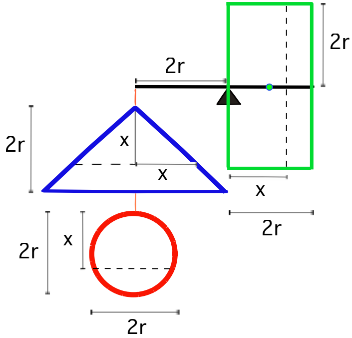
Cone CS: $+\pi x^2$;

Sphere CS: $\pi s^2 = 2\pi r x - \pi x^2$;

(Cone CS + Sphere CS) = $2\pi r x$ Cylinder CS: $4\pi r^2$

$2r$ · (Cone CS + Sphere CS) = $4\pi r^2 x$ = Cylinder CS · x **QED**

Method (Step 3): The Derivation



By previous slide: Each x-section of cylinder balances the combined x-sections of the hanging cone and sphere. Therefore, **the volumes balance**: $2r \cdot (C_o + S) = r \cdot C_y$, so

$$C_o = \frac{8}{3}\pi r^3, \quad C_y = 8\pi r^3 \quad \Longrightarrow \quad S = \frac{4}{3}\pi r^3$$

Using modern terms, we would express this as:

$$2r(C_o + S) =$$

$$\int_0^{2r} (2r)(2\pi rx)dx = 8\pi r^4 = \int_0^{2r} (x)(4\pi r^2)dx$$

$$= rC_y$$

Archimedes did this nearly 2 millenia before Newton and Leibniz.

Archimedes' Proudest Achievement

Archimedes asked that a cylinder and sphere be mounted on his tomb, displaying their proportional volumes.

The sphere is $2/3$ the volume of a circumscribed cylinder:

$$S = \frac{4}{3}\pi r^3.$$

"Cicero discovering the Tomb of Archimedes – 1"

(Anon., 1806)

Frontispiece of Weinzierl's German translation of Cicero's *Tusculan Disputations*

"Cicero discovering the Tomb of Archimedes – 2"

(Martin Knoller (1725-1804))

“Cicero discovering the Tomb of Archimedes – 3”

(Benjamin West (1738-1820))

"Cicero discovering the Tomb of Archimedes – 4"

(Hubert Robert (1733-1808))

"Ciceron Decouvrant le Tombeau d'Archimede" – 5

(Pierre Henri de Valenciennes (1750-1819))

Conclusion

Archimedes invented physical modeling, using rigorous mathematical deductions from specified physical axioms.

He formulated and *proved* the Law of the Lever, based on center-of-mass.

He anticipated methods of integral calculus: Cavalieri's Principle and Fubini's theorem.

And now we know his Mechanical Method, by which he applied the Law of the Lever to determine the volume of a sphere.

Recommended reading

- Asger Aaboe, *Episodes from the early history of mathematics*, Mathematical Association of America, 1998.
- Archimedes, *The works of Archimedes*, Dover Publications, (Thomas L. Heath edition), 2002.
- E. J. Dijksterhuis, *Archimedes*, Acta Historica Scientiarum Naturalium et Medicinalium, vol 12, 1956.
- Laubenbacher and Pengelley, *Mathematical Expeditions — Chronicles by the Explorers*, Springer, 1999.
- Sherman Stein, *Archimedes: What Did He Do Besides Cry Eureka?* Mathematical Association of America, 1999.

- Nietz & Noel, *The Archimedes Codex: How a Medieval Prayer Book Is Revealing the True Genius of Antiquity's Greatest Scientist*, Da Capo Press, Jan 9, 2009.
- The Archimedes Palimpsest ⟨`http://www.archimedespalimpsest.org/`⟩

In the Endnotes, the slides continue with:

An alternative discussion of the Law of the Lever using the concept of *torque*,

A sketch of Archimedes' derivation of the surface area of a disk (πr^2) and a sphere (πD^2) and

Brief commentary.

A text version with more details will be available on my web.

These slides are there, too:

www.mikeraugh.org

Contact me at,

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(Supplementary sections follow.)

Endnote 1

Torque is defined and determined using a differential equation based on the Axiom of Equivalence.

The Law of the Lever is inferred.

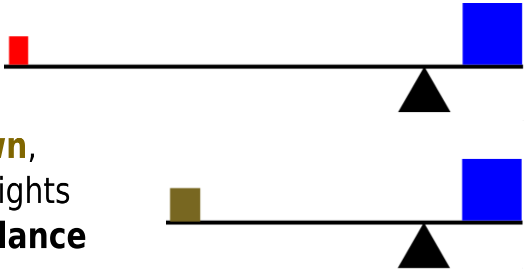
Concludes with discussion of “action at a distance” compared to “argument from cause”, with compound pulley as example.

First, Some Balance Axioms

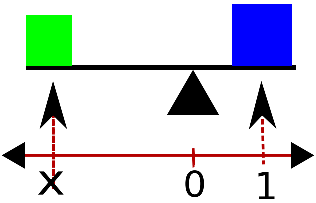
These axioms are assumed in the following derivations. They are not the same as Archimedes' axioms:

1. (Basic) Equal masses at equal distances are balanced.
2. (Symmetry) Balanced masses remain balanced under reflection about the fulcrum.
3. (Linearity) A balanced set of masses added to (or subracted from) a balanced set of masses remains balanced.
4. (Equivalence) A distributed mass balances like the same mass concentrated at the center-of-mass.

We begin with a graphic depiction of "torque."

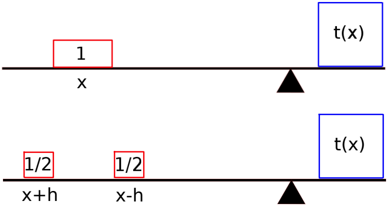


The red, **brown**, and **green** weights all **counterbalance** the same **blue** weight. They exert the same "**torque**", **defined** as the **blue** weight.



We compute "torque" by solving a simple differential equation.

Let $t(x)$ be the torque exerted by 1 unit of weight at a distance of x units from the fulcrum. By Axiom of Equivalence:



$$\frac{t(x+h)+t(x-h)}{2} = t(x) \implies$$

$$\lim_{h\to 0} \frac{t(x+h)-2t(x)+t(x-h)}{h^2} = \frac{d^2}{dx^2}t(x) = 0$$

$$\implies t(x) = ax+b = x$$

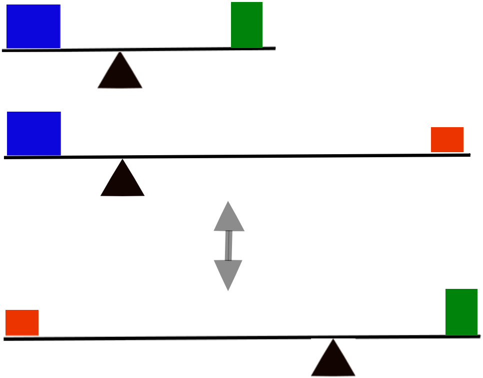
Now we have a formula for “big-T” Torque.

By the Axiom of Linearity the torque exerted by an arbitrary force F at a distance x from the fulcrum is $F \cdot t(x) = F \cdot x$

A force of weight W applied at a distance x from the fulcrum exerts a torque of

$$T(W, x) = W \cdot t(x) = W \cdot x$$

Law of the Lever: A Proof with One Word



$$W_1 \cdot x_1 = Torque = W_2 \cdot x_2$$

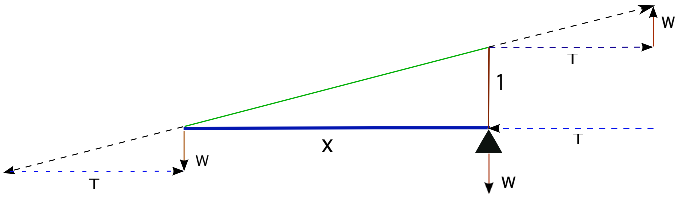
"Action at a Distance" vs "Argument from Cause"

Archimedes avoided specifying cause; he did not explain how torque is transmitted through a lever. Instead he assumed the mutual influence of objects balanced on a beam according to the empirical Axiom of Equivalence, as likewise Newton based gravitation theory on observed behavior, avoiding hypotheses about the cause of attraction. Let's call this kind of unexplained force "action at a distance."

We can incorporate cause into the derivation of torque by using the parallelogram law for composition of forces, as outlined in the next slide. The parallelogram law is itself an empirically derived axiom, developed by Stevins, Roberval and Newton. (Rene Dugas, "A History of Mechanics," Dover Edition, 1988.)

Consider the beam as a material object that transmits pressure and tension. In the next slide, a triangle is used as example, wherein the hypotenuse (think wire) transmits tension, and the lower longitudinal leg of the triangle (think rod) transmits pressure. Torque $T = W \cdot x$ above the fulcrum results from loading the beam with force W.

Archimedes Assumed Action at a Distance, But a Causal Proof is Possible



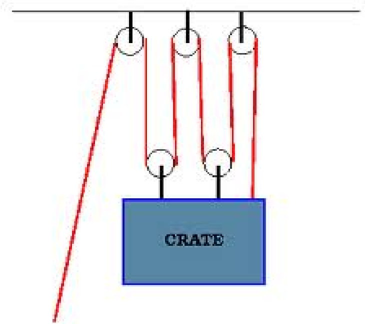
Suppose lever is triangular, loaded with weight W at lever's end.

Resolve force (W), where torque component is T. By similar triangles: $T = W \cdot x$

Note that the downward load at fulcrum is also W.

The same argument generalizes to a beam constructed as a truss, where x then is the length of the truss.

Compound Pulley: Archimedes' Other Form of "Leverage"



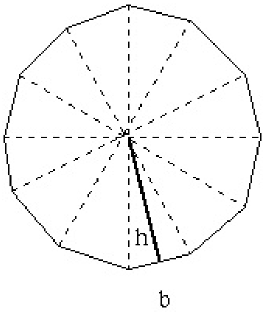
Assuming tension in rope is uniform, mechanical advantage is counted by number of segments pulling on load: 5 times in left panel, 6 times on right (at a cost of having to pull that much farther). Here the cause is self-evident: force transmitted through rope.

Endnote 2

Archimedes’ derivation of the area of a disk is sketched.

His analogous argument for determing the surface of a sphere is described.

Surface Area of a Sphere — First the Disk



(UBC Calculus Online, www.ugrad.math.ubc.ca)

Archimedes deduced disk Area as limit of n inscribed triangles:

$$(\text{as } n \to \infty) \quad h \to r, \quad \text{and } n\left(\frac{1}{2}h\frac{2\pi r}{n}\right) \to \pi r^2 = \text{Area}$$

Surface Area of a Sphere — Archimedes' Idea

Same idea as for the disk:

Imagine a sphere as comprised of many thin cones with apexes at the center and bases at the surface. Say, a spherical planet platted with one-acre plots; the surface area would equal the number of plots.

Summing the conical volumes, in the limit they amount to a cone with base equal to the surface of the sphere S and height equal to the radius r.

$$\frac{4}{3}\pi r^3 = \frac{1}{3} rS \quad \to \quad S = 4\pi r^2$$

Endnote 3

Brief Comments Interpolated in Talk

Quality of Archimedes' Work

"In weightiness of matter and elegance of style, no classsical mathematics treatise surpasses the works of Archimedes. This was recognized in antiquity; thus Plutarch says of Archimedes' works:

> *'It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his genius; while others think that incredible effort and toil produced these, to all appearances, easy and unlaboured results.'* "

(Aaboe in *Recommended reading*)

"Archimedes is so clever that sometimes I think that if you want an example of someone brought from outer space it would be Archimedes. Because he, in my view, is so original and so imaginative that I think he is better than Newton. Whereas Newton said, 'I have only seen so far because I have been standing on the shoulders of other giants,' there was nobody for Archimedes, nobody's shoulders for Archimedes to stand on. He is the first physicist and the first applied mathematician. And he did it all on his own from nowhere."

(Lewis Wolpert in *On Shoulders of Giants* by Melvyn Bragg, 1998)

Archimedes' "Method of Mechanical Theorems."

Archimedes' geometric proof for the volume of a sphere was well known, but the method by which he discovered the result remained a mystery until 1906.

The *Archimedes Palimpsest*, in which Archimedes described his method, was found in Istanbul in a Byzantine crypt, then lost and recovered again in 1998.

The Palimpsest contained a tenth-century copy of a Greek MS that was scraped, washed and overlaid with Christian liturgy and other writings.

(See *Recommended reading*).

The Roman orator Cicero found the Archimedes' tomb.

Archimedes was killed in 212 BC.

Archimedes had asked that a cylinder and sphere be mounted on his tomb, displaying their proportional volumes: 2/3.

The tomb was built and lost until the figures of the cylinder and the sphere enabled Cicero to find it 137 years later.

Cicero found the tomb in 75 BC. He wrote: "So one of the most

famous cities in the Greek world would have remained in total ignorance of the tomb of the most brilliant citizen it had ever produced, had a man from Arpinum not come and pointed it out!"

(Aaboe in *Recommended reading*)