

*Leibniz used Calculus to solve the Catenary
Problem:
But he presented it as a Euclidean Construction
without Explanation.*

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RIPS at IPAM

June 30, 2017

Catenary: Derived from Latin Word for Chain, Catena



***“Pink Moon”**: Full Moon April 2017*



Boston Harbor (mrr)

Some History

1638, Galileo discussed the hanging-chain problem.

1690, Jacob Bernoulli published a challenge to solve the problem.

1691, Leibniz and Johann Bernoulli published the first solutions.

1761, Johann Heinrich Lambert introduced hyperbolic functions and named them:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Descartes: A curve must be defined by a construction.

He deemed no other method was sufficiently accurate.

Descartes' rule was disputed during the late 1600's.

Analysts were beginning to use calculus and power series.

They were led by Newton and Leibniz.

Leibniz played it both ways.

Leibniz presented the solution as a Euclidean construction.

But he did not publish the derivation.

He explained it in a private letter.

The construction turns out to be impossible!

A paradox?

And so our story begins....

Analytic Formulation of the Catenary

In modern terms (with scaling factor a),

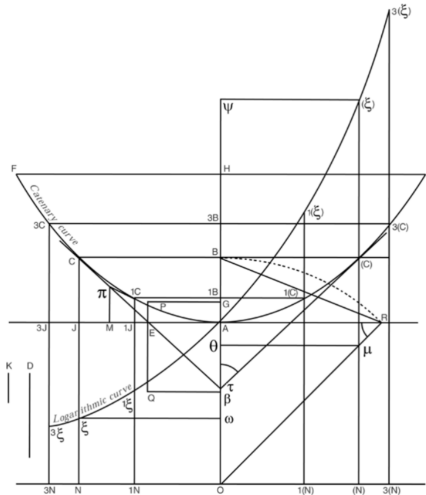
$$\frac{y}{a} = \cosh \frac{x}{a}.$$

Or,

$$y = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}.$$

The curve is **bilaterally symmetric** about the y -axis,
and **the lowest point is at $(0, a)$** .

Leibniz's Representation of the Catenary: Draughted using a *Ruler & Compass*



The segments D and K are assumed given.

Leibniz uses only their **ratio**: $\frac{d}{k}$.

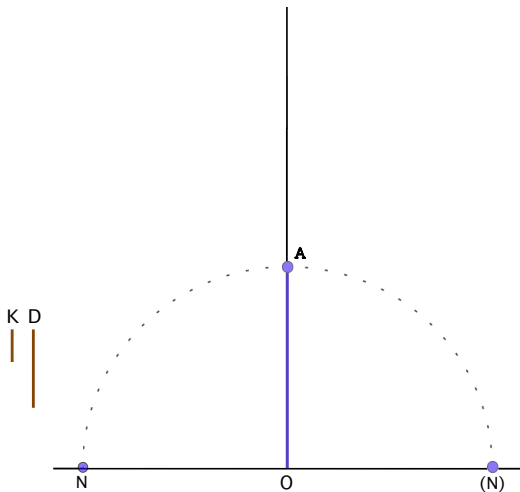
If the ratio is not constructable, then neither is the curve.

But D and K are **given**, so their ratio could be **anything**.

This fact can make a **fictitious “construction”** correct,
(in theory).

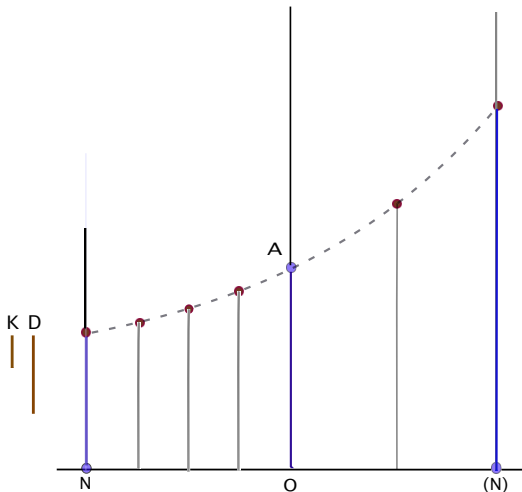
This resolves the paradox for Analysts but **not for Cartesians**.

First Steps of the Construction



Draw: (1) **horizontal axis**, (2) **origin O** and **vertical axis**;
(3) choose **$\overline{OA}$** as **unit**, (4) mark **unit lengths on horizontal axis**.

Constructing the “Logarithmic Curve”



Ordinates over N & O and O & (N) are in ratio $K:D$.
Middling ordinates are determined by geometric means.

The “Logarithmic Curve” in Cartesian Coordinates (Represented as an Exponential Curve)

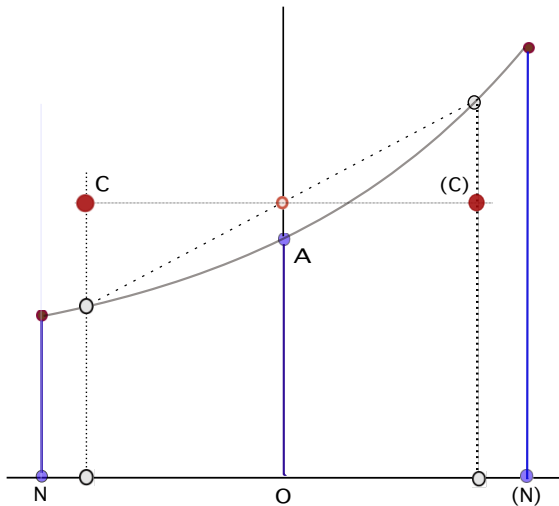
Given two points (x_1, y_1) and (x_2, y_2) , get a new one:

$$\left(\frac{x_1 + x_2}{2}, \sqrt{y_1 y_2} \right)$$

The construction yields dense points on the curve,

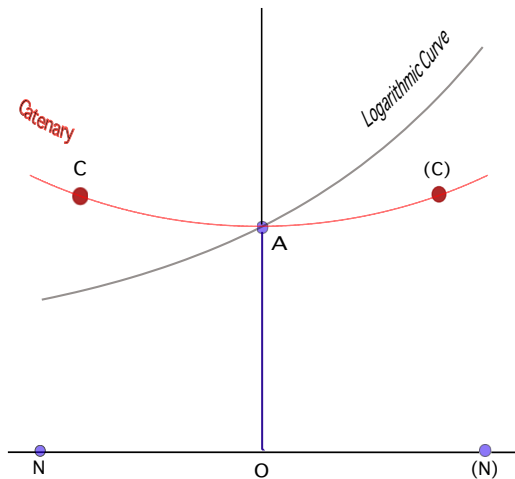
$$y(x) = a \left(\frac{d}{k} \right)^{x/a} \quad (x \text{ a binary number})$$

Construction of the “Catenary”



As constructed: $C(x) = \frac{r^x + r^{-x}}{2}$, with $a = 1$ and $r = \frac{d}{k}$

Leibniz's "Catenary" is Built on an Exponential Curve.



$$z(x) = \frac{a}{2} \cdot \left\{ \left(\frac{d}{k} \right)^{\frac{x}{a}} + \left(\frac{d}{k} \right)^{-\frac{x}{a}} \right\}$$

Is Leibniz's “catenary” truly a catenary?

A true catenary must be of the form:

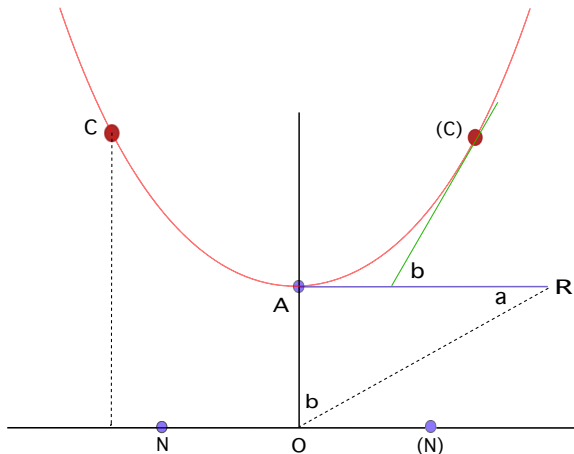
$$z(x) = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$$

Leibniz needed the ratio $d/k = e$.

In effect, he used that — as revealed in his figure.

So, as we shall see, it is a true catenary.

Two Examples Requiring a True Catenary: $d/k = e$

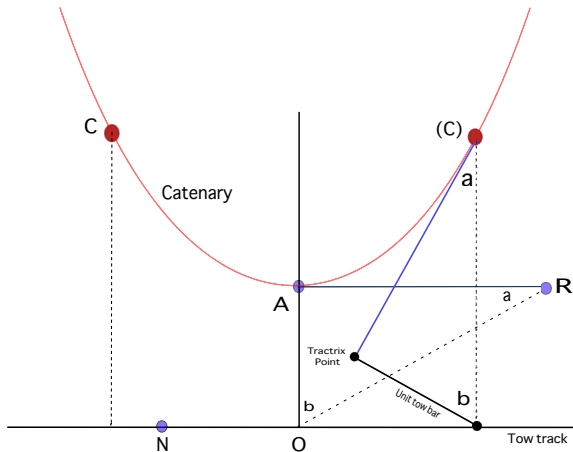


Segment $\overline{AR}$ is equal in length to $\text{arc } \widehat{CA}$.

Tangent at (C) follows from fact that $\angle b$ is the complement of $\angle a$.

$$(y = \cosh x)$$

For Fun: The Tractrix is the Involute of the Catenary.



Rotate the arc-length triangle to trace a tractrix.
(A problem solved by Leibniz later, not in his figure.)

How did Leibniz arrive at his solution?

He explained his derivation in a letter:

To **Rudolf Christian Von Bodenhausen**, August 1691, with attached Latin text, “Analysis problematis catenarii”, in G. W. Leibniz, *Sämtliche Schriften und Briefe*, series III, volume 5 (2003), p. 143-155

(Thanks to Siegmund Probst)

The Derivation Part 1:

Leibniz deduced, as did Bernoulli (see Ferguson), that:

$$\frac{dy}{dx} = y' = \frac{s(x)}{a} \implies dx = \frac{a \, dy}{\sqrt{y^2 + 2ay}}$$

(s = arc length, a = scaling factor)

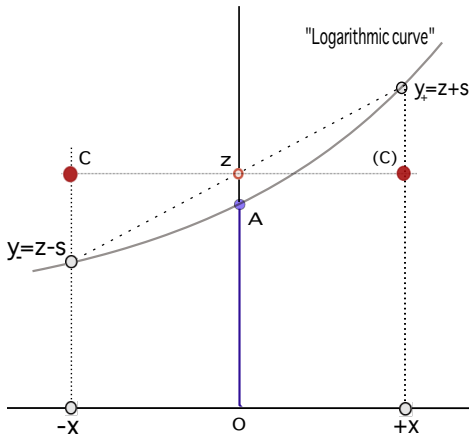
Setting $z = y + a$, Leibniz inferred $s = \sqrt{z^2 - a^2}$

$$\text{Or, } z^2 - s^2 = a^2.$$

From this, **the leading clue**:

$$(\text{We can use } a = 1): \quad (z - s)(z + s) = 1.$$

A solution is suggested by this graph:



$$(z - s)(z + s) = 1$$

Let: $-x = \ln(z - s), +x = \ln(z + s).$

Then: $z = \frac{y_- + y_+}{2}$ (Why not $\log_b, b \neq e$?)

The Derivation Part 2

$$(z - s)(z + s) = 1 \text{ suggests: } \ln(z - s) = -\ln(z + s).$$

Leibniz approach: $\omega(x) = z - s$ and use $d(\ln \omega)$:

$$\frac{d\omega}{\omega} = \frac{dz - ds}{z - s}$$

$$\text{But } z dz = s ds \text{ and } dz = s dx \implies$$

$$\frac{d\omega}{\omega} = \frac{dz - z dx}{z - dz/dx} = -dx \implies$$

$$-x = \ln(z - s), \quad \text{similarly} \quad +x = \ln(z + s)$$

$$[\text{ICs } z(0) = 1, z'(0) = 0 \implies \text{Integration constant} = 0.]$$

*The Derivation **Part 2b**: Bring back “a”*

Leibniz (simplified): $\frac{(z-s)}{a} \frac{(z+s)}{a} = 1$; **let:** $\omega = \frac{(z-s)}{a}$.

$$\frac{d\omega}{\omega} = \frac{dz - \textcolor{blue}{ds}}{z - \textcolor{red}{s}} = -\frac{dx}{a}$$

So,

$$-\frac{x}{a} = \ln \frac{(z-s)}{a}, \quad \text{similarly} \quad +\frac{x}{a} = \ln \frac{(z+s)}{a}$$

$$\frac{\textcolor{red}{z}}{\textcolor{red}{a}} = \frac{\textcolor{red}{e^{\frac{x}{a}}} + \textcolor{red}{e^{-\frac{x}{a}}}}{\textcolor{red}{2}} = \cosh \frac{x}{a}$$

[ICs $z(0) = a, z'(0) = 0 \implies$ Integration constant = 0.]

*“Let those who don’t know the new analysis
try their luck!”*

To Bodenhause Leibniz also wrote that
he left out one specification for his construction:

!!! D and K were in ratio 1 to 2.7182818 !!!

Why this ostentatious approximation?

He didn’t name it or explain it,
and it was too precise to use by the draughtsman.

He had already built-in the exact value for e using $\frac{d\omega}{\omega}$.

A Sample of Leibniz's Technique: Differentials I

Prove: $\frac{dy}{dx} = s \implies dx = \frac{dy}{\sqrt{y^2 + 2y}}$

Use: $(ds)^2 = (dx)^2 + (dy)^2$

Let dx be constant, differentiate:

$$ddy = ds \, dx \quad \text{and} \quad ds \, dds = dy \, ddy$$

Combine (and “anti-differentiate”):

$$dds = dy \, dx \implies ds = y \, dx + c \, dx \implies$$

$$(dx)^2 + (dy)^2 = (y^2 + 2y + c)(dx)^2$$

A Quick Sample of Technique: Differentials II

$$(dx)^2 + (dy)^2 = (y^2 + 2y + c)(dx)^2 \implies$$

[Use initial conditions, $y(0) = y'(0) = 0$]

$$(dy)^2 = (y^2 + 2y)(dx)^2$$

$$\frac{dy}{dx} = \sqrt{y^2 + 2y} \implies dx = \frac{dy}{\sqrt{y^2 + 2y}}$$

QED

Conclusion

In 1761 Lambert named the “Hyperbolic Cosine”:

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

In 1691 Leibniz had already called it the Catenary!

At the time of Leibniz, the Cartesian canon of construction began yielding to the methods of calculus.

Leibniz used conventional constructions to exhibit curves, but he relied on analysis as well.

Acknowledgments

Communications about Leibniz translations and literature:

Eberhard Knobloch, Berlin-Brandenburg Academy of Science

Siegmond Probst, Leibniz Archive, Göttingen Science Academy

Discussions about the use of differentials in early calculus:

Robert Bradley, Adelphi University

Discussion about the tractrix:

Jorge Balbás, California State University, IPAM

References I

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Bos, Henk , *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*, Springer, 2001.

Guicciardini, Niccolo, *Isaac Newton on Mathematical Certainty and Method*, MIT Press, 2011.

Hofmann, Joseph, *Leibniz in Paris 1672 – 1676, His Growth to Mathematical Maturity*, Cambridge University Press, 1974.

Leibniz, Gottfried Wilhelm, *Die Mathematischen Zeitschriftenartikel* (German translation and comments by Hess und Babin), Georg Olms Verlag, 2011. (Also see Ferguson's English translation on Internet.)

Wikipedia and Internet used for dates and graphics.

References II

For a scholarly treatment of methods and notation used by the earliest followers of Leibniz, see,

L'hôpital's analyse des infiniments petits: an annotated translation with source material by Johann Bernoulli, edited by Robert E. Bradley et al, Birkhauser 2015, (pp 62-65 for use of dx as free variable).

For Euclidean constructions and constructible numbers, see

John M. Lee, *Axiomatic Geometry*, American Mathematical Society, 2012.

For some of Leibniz's own earliest work from a MS written in 1676 during his years in France, see,

G. W. Leibniz, *quadratura arithmetica circuli, ellipseos et hyperbolae*, Herausgegeben und mit einem Nachwort versehen von Eberhard Knobloch, aus dem Lateinischen übersetzt von Otto Hamborg, Springer Spektrum 2016.

A Monumental Catenary Arch 631 Feet High



The Gateway Arch, St. Louis, Missouri

Thanks for your attention.

Supplementary notes (and these slides) available at,

www.mikeraugh.org

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