How did Leibniz Solve the Catenary Problem? A Mystery Story

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Huygens named a hanging chain the Catenary and proved the shape is not a parabola:

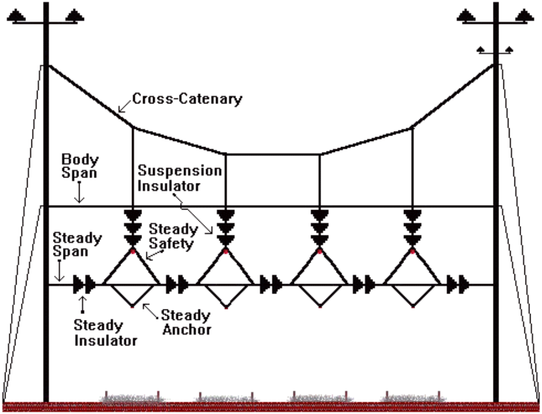
Derived from the Latin word for chain: *catena*

Internet

High-tension Power Lines

Internet, QT Luong

Supporting Catenary for a Railroad Power Line



Internet, Steve Sconfienza, Ph.D.

Another supporting Catenary: a "Hawser"

This Hawser is looped over a Dock Bollard.

Internet, wiseGeek

Catenaries posed a challenging physics problem.

Galileo mentioned the problem in 1638. It was solved in the late 1600's using the new methods of calculus.

Liebniz was interested in calculating machines.
He gave a *compass & straightedge construction* of a catenary and explained how a real one could be used for finding logarithms.

Johann Bernoulli advocated using differential equations to formulate physics problems.
His solution showcased his approach.

They solved the problem in response to a published challenge.

1690 Jacob Bernoulli published the Challenge:

Determine the curve of a freely hanging chain!

Jacob Bernoulli (1665–1705)

Internet

1691 Leibniz and Johann Bernoulli published solutions.

G. W. Leibniz (1646–1716)

Johann Bernoulli (1667–1748)

Internet

Leibniz's construction was a classic Euclidean "Ruler & Compass" construction.

Nobody knows how Leibniz arrived at his construction.

He never specified the underlying function nor gave it a name.

Manifestly, it was based on expert knowledge of the exponential function and hyperbolic cosine.

But the hyperbolic cosine wasn't known by a specific expression or name until 70 years later (Lambert 1761)!!!

Bernoulli used the new methods of differential calculus to derive a differential equation for the Catenary.

He wrote the correct equation without solving it. He used it to imply that Leibniz's construction was correct:

$$dy = \frac{a\,dx}{\sqrt{2ax + x^2}}$$

We can solve this equation ourselves:

$$y(x) = a \cdot \cosh\left(\frac{x}{a}\right) - a$$

(A catenary pulled down to the Origin)

Oddly, Leibniz's construction wasn't what Bernoulli found!

Leibniz should have constructed a curve equivalent to the solution of Bernoulli's equation:

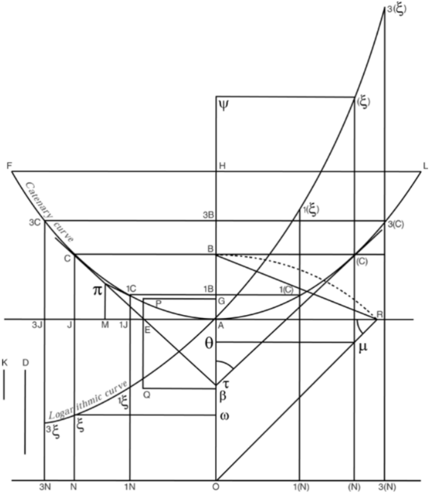
$$y(x) = a \cdot \cosh \frac{x}{a} = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2} \quad (a \in \mathbb{R})$$

But he didn't!

(Bernoulli used the parameter "a" correctly as the length of the "subtangent" of a logarithmic curve; Leibniz did not.)

And so our story begins....

Figure in Leibniz Article illustrating his Ruler & Compass Construction



Leibniz begins with segments K and D, and one horizontal line.

A classic construction begins with a set of randomly placed points and random segments: (1) a straightedge (ruler without distance marks) can be used to draw a line determined by two given points; (2) a compass can be used to draw random circular arcs or of a radius determined by a given segment, or to mark points on a line .

From these initial elements, all other points and lines are constructed using a straightedge and a compass, illustrated in the figure.

In the figure, a horizontal line through point O is given. Leibniz constructs a line perpendicular at O, and on it sets an arbitrary point A (defining segment $\overline{\mathrm{OA}}$), through which a parallel to the first line is constructed. Segments K and D of unspecified length are shown to the left of the figure, D longer than K. I use a, k and d to denote lengths of the three segments.

Leibniz proceeds by constructing segments $\overline{\mathrm{ON}}$ and $\overline{\mathrm{O(N)}}$ equal to $\overline{\mathrm{OA}}$ with perpendiculars above each, then $\overline{\mathrm{N}\xi}$ of length ak/d and $\overline{1(\mathrm{N})1(\xi)}$ of length ad/k. Leibniz achieves these proportions by construction, not by the numbers I have used. He identifies $\overline{\mathrm{OA}}$ as his unit, so I use $a = 1$ where appropriate but retain an undetermined a for generality.

Leibniz's Rules for Constructing his "Logarithmic Curve", represented in Cartesian Coordinates

I use numerical coordinates with origin O at $(0,0)$, points N and ξ at $(-a,0)$ and $(-a, ak/d)$, and (N) and (ξ) at $(+a,0)$ and $(+a, ad/k)$, and so forth to tabulate the succeeding points *constructed* by Leibniz.

Given two points on the curve, he constructs the geometric mean of their ordinates then sets his new point halfway between:

$$(x_1, y_1) \quad \text{and} \quad (x_2, y_2) \quad \to \quad \left(\frac{x_1 + x_2}{2}, \sqrt{y_1 y_2}\right)$$

Repeating results for "binary divisions" of any constructed interval yields dense points on a curve of type,

$$y(x) = ar^x, \quad (a \text{ is given, } r \text{ can be determined})$$

It is an important distinction that this equation is a cartesian representation of a curve *constructed* by Leibniz, *not graphed* by using cartesian coordinates.

We view the "logarithmic curve" as an exponential curve:

The initial conditions require that $r = d/k$, and therefore:

$$y(x) = a \cdot \left(\frac{d}{k}\right)^{\frac{x}{a}}$$

The numbers x represent constructable points on Leibniz's horizontal line through O, and the corresponding numbers representing ordinates are also constructable in the way explained by Leibniz.

(Remember: the segments D and K were given without specified lengths. But if they are presumed to have specified lengths they must be constructable, otherwise the curve defined by Leibniz will *not* be constructable *except in theory*. We shall see that Leibniz does presume a non-constructable ratio for them: *e*, not constructable because it is a transcendental number.)

Leibniz used $y(x)$ and $y(-x)$ to obtain his "catenary":

$$z(x) = \frac{a}{2} \cdot \left\{ \left(\frac{d}{k}\right)^{\frac{x}{a}} + \left(\frac{d}{k}\right)^{-\frac{x}{a}} \right\}$$

He claimed his construction would yield a catenary.

But a catenary must be of the form:

$$z(x) = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$$

Leibniz had to assign the ratio $d/k = e$ to obtain a catenary!

Internal evidence shows that he knew this.

Leibniz had already set $a = 1$ for his unit,

and he had to use $d/k = e$ to get a catenary:

$$y = \frac{e^x + e^{-x}}{2} = \cosh x$$

This is the only possibility for his claims about his construction to be correct: for example, his construction of a tangent to the curve, and his specification of a segment of equal length to a portion of the curve (more about this in “Afterthoughts”).

How he came to all of this is a mystery!

Because....

Ideas published after the Leibniz construction of 1691:

Lambert: 1761, introduced hyperbolic functions and named them —

seventy years after Leibniz!!

Euler: 1748, published first "modern" calculus text, *Introductio in analysin infinitorum* ,

presenting the first integrated theory of the natural log and exponential functions

But what was in the background for Leibniz?

Napier: 1614, Table of natural logarithm

Fermat: $\sim$ 1656, $\int x^n\,dx = x^{n+1}/(n+1)$, $(n \neq -1)$

Wallis: $\sim$ 1658, $x^{p/q}$, "infinite methods" like,

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots\cdots\cdots$$

Saint-Vincent: 1647, $\int \frac{dx}{x} = \text{logarithm}$

And these also:

Nicholas Mercator: 1668, Series for the natural logarithm,

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Newton: 1676, Wrote to Leibniz about infinite series, including something Leibniz apparently already knew,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So we are left with these mysteries:

How did Leibniz infer the form of the catenary?

Leibniz scholars have supposed he deduced the catenary by assuming it was the curve with deepest center of gravity.

Leibniz left a hint to that effect.

I have found no more evidence of variational technics in his writing than for the theory of the exponential function.

But logarithms and exponential functions were "in the air," and Leibniz exploits them explicitly in his construction.

Some Afterthoughts

I offer below some ideas about how Leibniz may have been led almost directly to the analysis that underlay his construction. I write loosely in a way that historians warn against: imposing present conceptions on actors of the past.

I use modern concepts and notation for functions and differentiation, when in fact Leibniz did not write in the same terms. He illustrated much of his early mathematics in geometric figures, common for mathematicians at the time. But he was breaking away from the standards of the day, so it is possible that although he wrote in one way, his thoughts were more advanced.

I will write more about this elsewhere to show how what I express below can be rewritten in a mathematical idiom close to what Leibniz used in his writings. I think the following ideas, when properly transcribed, would have been within an easy reach for him. They show that the catenary problem can be solved quite naturally by a route that exposes the logarithm and hyperbolic cosine as central to the solution.

An easy Route for Leibniz?

Leibniz would know, as Bernoulli did (see Ferguson), that:

$$\frac{dy}{dx} = \frac{s(x)}{c}, \quad \text{(required for equilibrium)}$$

[$s(x) =$ vertical weight below (x, y), $c =$ constant horizontal tension]

$$\text{Therefore:} \quad y'' = C\sqrt{1 + y'^2}, \quad (C = 1/c)$$

$$\Longrightarrow \quad w'^2 = C^2(1 + w^2), \quad (\text{use } w(x) = y'(x))$$

$$\Longrightarrow \quad w'w'' = C^2 ww' \quad \Longrightarrow \quad w'' = C^2 w$$

For Leibniz, $C = 1$

Leibniz would need to solve $w'' = w$

But he knew the power series for the exponential function:

$$E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and that $E' = E$

He could guess a power series $F(x)$ such that $F'' = F$.

Simply remove the even-numbered (or odd-numbered) terms from the series for E !!

Drop terms from the series for E to solve $F'' = F$:

$$F_1 = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{E(x) + E(-x)}{2} \equiv \cosh x,$$

$$F_2 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{E(x) - E(-x)}{2} \equiv \sinh x.$$

And so it follows that,

$$\frac{d \sinh x}{dx} = \cosh x, \qquad \frac{d \cosh x}{dx} = \sinh x.$$

And, because E is the exponential function (or by multiplying the series), $E(x)E(-x) = 1$ so that,

$$\cosh^2 x - \sinh^2 x = 1$$

Those results explain the constructions that Leibniz noted for the tangent and arc length:

The prescription Leibniz gave for constructing his tangent line implies:

$$y'(x) = \sqrt{y^2 - 1}$$

And arc length follows because,

$$s = \int \sqrt{1 + y'^2}\, dx = \int \cosh x \, dx = \sinh x = \sqrt{y^2 - 1}$$

(The Cartesian representation (x, y) represents Leibniz's point C on the catenary. Integration is over the interval from $(0, 1)$ to (x, y). To apply these results to the illustration of the construction, see the next page for references.)

Points of Interest in the Construction (see Illustration)

C = a typical point on the catenary.

$\overline{\text{OA}}$ is Leibniz's "unit", and

$\overline{\text{OA}} = \overline{\text{CN}} = \overline{\text{(C)(N)}}$.

Hypotenuse $\overline{\text{OR}} = \overline{\text{OB}} = \overline{\text{NC}}$.

The following facts can be verified from equations on the preceding page:

$\overline{\text{AR}} = \text{arc } \widehat{\text{AC}}$.

The tangent at (C) is determined by the complementary angles marked at R and τ.

One Last Question for You

In 1761 Lambert gave the name "Hyperbolic Cosine" to the function,

$$y = \frac{e^x + e^{-x}}{2}.$$

But can't we say that in 1691 Leibniz had already called it the Catenary?

References I

Barnett, Janet Heine, *Enter, stage center: The early drama of the hyperbolic functions*, MAA Mathematical Association of America (2004),

Bernoulli, Johann, *Lectures on the Integral Calculus, Part III*, (Ferguson Translation), pdf on Internet, 2004.

Guicciardini, Niccolo, *Isaac Newton on Mathematical Certainty and Method*, MIT Press, 2011.

Hoffman, Joseph, *Leibniz in Paris 1672 – 1676, His Growth to Mathematical Maturity*, Cambridge University Press, 1974.

Iliffe & Smith, eds., *Cambridge Companion to Newton*, Cambridge University Press, 2016(2002).

Leibniz, Gottfried Wilhelm, *Die Mathematischen Zeitschriftenartikel* (German translation and comments by Hess und Babin), Georg Olms Verlag, 2011. (Also see Ferguson's English translation on Internet.)

Wikipedia and Internet used for dates and graphics.

References II

For a scholarly treatment of methods and notation used by the earliest followers of Leibniz, see,

L'hôpital's analyse des infiniments petits: an annotated translation with source material by Johann Bernoulli, edited by Robert E. Bradley et al, Birkhauser 2015.

For a brisk reprise of Leibniz's explanation of how to use a catenary to determine logarithms, see,

Viktor Blåsjö, *How to Find the Logarithm of Any Number Using Nothing But a Piece of String*, vol 47(2) March 2016, The College Mathematics Journal.

For some of Leibniz's own earliest work from a MS written in 1676 during his years in France, see,

G. W. Leibniz, *quadratura arithmetica circuli, ellipseos et hyperbolae*, Herausgegeben und mit einem Nachwort versehen von Eberhard Knobloch, aus dem Lateinishen übersetzt von Otto Hamborg, Springer Spektrum 2016.

References III

If you are interested in Classical Geometry and Construction, you can find more about them in these references. For an excellent edition of Euclid, based on the famous Heath translation, see,

Euclid, *Euclid's Elements*, Green Lion Press, 2002.

Next is an advanced text for high-school teachers, covering Euclidean geometry and ruler-and-compass constructions. Why study Euclidean geometry in a modern setting like this? Read the preface:

John M. Lee, *Axiomatic Geometry*, American Mathematical Society, 2013.

For a more advanced college-level text covering Euclidean Geometry from a perspective of modern algebra, see,

Robin Hartshorne, *Geometry: Euclid and Beyond*, Springer 2010.

A Catenary Arch is Stable — Even one of Mud.

The compressive forces are in perfect alignment with the arch, as are the tensions in a hanging chain.

Internet, Maxcorradi

A Monumental Catenary Arch 631 Feet High

The Gateway Arch, St. Louis, Missouri

Internet

Acknowledgments

Conversations:
Pof. Adrian Rice

References to Leibniz translations and commentaries:
Prof. Eberhard Knobloch

Cautions about imposing modern ideas on historic actors:
Prof. Robert E. Bradley

These slides are on my web.

www.mikeraugh.org / LeibnizSlides

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