Real Numbers are Not Real: The Innumerable Infinities of Georg Cantor

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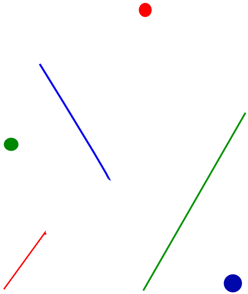
We model the world with images and concepts.

(Internet)

This is not Georg Cantor (1845-1918) — it's an image.

But these are real points and lines, not just images or mathematical idealizations:



We usually mean something like these — visible, inscribed on a thing or illustrated as here, with physical presence in some medium.

So, what are the "Real" Numbers?

We use them to represent points. But are they "out there", on a real line? Or are they just conceptual, imaginary?

Let's investigate! How many 9s can follow the decimal point?

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We could go on, but let's stop here.

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That's 1250 "nines".

Seems a lot, but it's not many in the Grand Scheme of Things.

The number represented by all those 9s is equal to

$$0.(9)_{1250} = 1 - 10^{-1250}$$

That's not equal to 1. We can get closer.

How far do you want to go?

Want to try $10^{1,834,097}$*?*

It would take that many 9s to fill all possible 410-page volumes of the fictional *Library of Babel*,** representing the number,

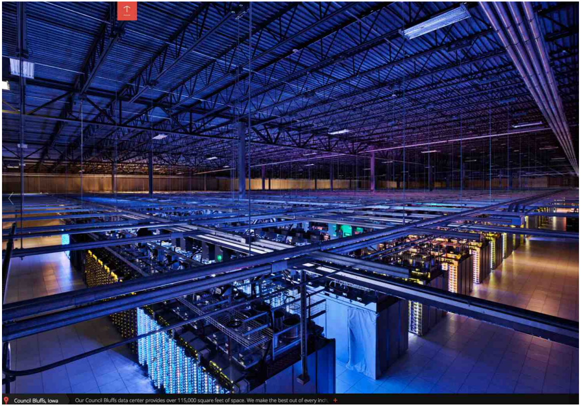
$$0.(9)_{10^{1,834,097}} = 1 - 10^{-10^{1,834,097}} \neq 1$$

Borges' Library would fill our known Universe more than $10^{1,834,013}$ times, but still not contain enough 9s to reach

EXACTLY 1

(**About Borges' Library, see references.)

Google data center can't hold an infinite number of 9s.



(Google)

1 Googol = 10^{100}, far beyond the reach of Google and trivial compared to the multi-universe-filling capacity of the Library of Babel.

NSA Data Center can't hold an infinite number of 9s.

1-million-square-foot data storehouse, capable of storing a yottabyte (10^{15} *Gigabytes*) of information, about 500 quintillion ($50 \cdot 10^{19}$) pages of text. Trivial compared to Babel Library.

(Cryptome Archive, Internet)

How many 9s are needed to reach exactly 1?

How many books full of 9s?

How many Universes packed with 9s?

ANSWER: You can’t get there with a finite number of 9s!

But there are not an infinity of 9s in the real world!

Nevertheless, Cantor took an imaginative leap and based his theory of real numbers on sets of infinite size.

Cantor strayed beyond the bounds of orthodoxy.

The great Gauss had outlawed infinite sets!!

He allowed infinity only in the sense of approaching a limit. Can you blame him?

Cantor would suffer for his transgression but finally triumph — as we shall see....

Why did he dare to do that?

Mathematicians understood that power series converge on intervals, for example:

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}, \quad |x| < 1$$

But trigonometric series had puzzling convergence properties:

$$f(x) = \lim_{n\to\infty} \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \sin kx + b_k \cos kx)$$

Cantor wanted to know: How many values of x could you ignore** so that $f(x) = 0$ still implies that all a's and b's $= 0$?

(**These are called *exceptional sets for uniqueness*.)

*Cantor needed a rigorous definition of real numbers and a theory of sets.***

Cantor had to specify not just the number of points that could be ignored, but also their structure , i.e. relative positions.

He defined the "real numbers", discovered "derived sets", and found that derived sets could be arranged in transfinite order:

$$1 \rightarrow \omega_0 \rightarrow \omega_1 \rightarrow \omega_2, \cdots \text{ (ordinal numbers)}$$

And he classified sets of infinitely different sizes:

$$\aleph_0, \aleph_1, \aleph_2, \cdots \text{ (cardinal numbers)}$$

(**See references for details not needed for the talk.)

So Cantor postulated infinite sets. Here are some results:

The set of positive integers is infinite and of cardinality (size) $\aleph_0$.

This can be written

$$|\mathbb{N}_{>0}| = |\{1, 2, 3, \cdots\}| = \aleph_0.$$

Cantor showed that there are more real numbers than there are integers, with consequences shocking to contemporaries.

First let's see why there are $\aleph_0$ even integers (why not fewer) and $\aleph_0$ rational numbers (why not more)!

Definition: Two sets have the same cardinality if they can be put in one-to-one correspondence.

The set of positive integers and the set of even integers have the same cardinality:

$$\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ 2 & \longleftrightarrow & 4 \\ 3 & \longleftrightarrow & 6 \\ 4 & \longleftrightarrow & 8 \\ & \vdots & \end{array}$$

Any set that can be put in a 1-to-1 relationship with the positive integers is said to be denumerable or countable.

The Rational Numbers are denumerable.

$$\begin{array}{ccccccc} 1 & \longleftrightarrow & 1 & \quad & 7 & \longleftrightarrow & \frac{1}{5} \\ 2 & \longleftrightarrow & \frac{1}{2} & & 8 & \longleftrightarrow & \frac{2}{5} \\ 3 & \longleftrightarrow & \frac{1}{3} & & 9 & \longleftrightarrow & \frac{3}{5} \\ 4 & \longleftrightarrow & \frac{2}{3} & & 10 & \longleftrightarrow & \frac{4}{5} \\ 5 & \longleftrightarrow & \frac{1}{4} & & 11 & \longleftrightarrow & \frac{1}{6} \\ 6 & \longleftrightarrow & \frac{3}{4} & & 12 & \longleftrightarrow & \frac{5}{6} \\ & & & \vdots & & & \end{array}$$

But the Real Numbers are Not denumerable!

They are a MUCH LARGER set.

Because, suppose the Real Numbers were denumerable.

(It is sufficient to consider just the unit interval $\mathbb{I} = [0, 1] \subset \mathbb{R}$.)

Then their decimal expansions — which Cantor supposed to be infinitely long — could be listed in sequential order:

$$\begin{array}{ccc} 1 & \longleftrightarrow & .a_1 a_2 a_3 a_4 a_5 \cdots \\ 2 & \longleftrightarrow & .b_1 b_2 b_3 b_4 b_5 \cdots \\ 3 & \longleftrightarrow & .c_1 c_2 c_3 c_4 c_5 \cdots \\ \vdots & & \end{array}$$

Cantor's *Diagonal argument*: this number can't be in the list:

$$.\overline{a_1 b_2 c_3 d_4 e_5 \cdots}$$

(The *overbar* means "change each digit to anything but "9".)

So the Real Numbers are not denumerable.

And therefore:

Infinity comes in different sizes: $|\mathbb{R}| = |\mathbb{I}| = 2^{\aleph_0} > \aleph_0$.**

Since the set of all names and algorithms are denumerable, it's impossible to specify most numbers.

Most real numbers are not computable!

So what are all those anonymous "real numbers" doing out there among the "knowable" ones?

(**"$2^{\aleph_0}$" is conventional notation for the cardinality of the continuum — see references for explanation.)

Noncomputable numbers were tailor-made for a purpose.

The set of real numbers was constructed to justify the theory of limits:

They were invented to make analysis logically rigorous. For this you need the noncomputable numbers as well as the computable ones.

They make it meaningful to speak of a computable number with a name, like π, for which we can never enumerate all the digits but can specify by a series like,

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{(Euler, 1735)}$$

But there is a rumple in Cantor's well-tailored theory.

Cantor's theory of the continuum (real number system) placed analysis on a rigorous foundation, but it also introduced a deep unresolved problem.

Cantor's Continuum Hypothesis:**

$$\aleph_1 = 2^{\aleph_0}$$

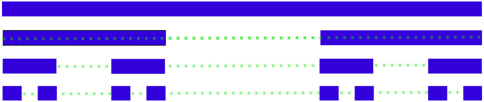
The axioms of set theory allow this (Gödel's theorem) but can't prove it (Cohen's theorem).

How can that be? Surely CH is probably false!? (Cohen)

(**$\aleph_1$ = cardinality of the set of countable ordinals, $2^{\aleph_0}$ = cardinality of the continuum.)

The Cantor Ternary Set, A Strange Thing — Part 1

Cantor showed how to remove practically all the points of the unit interval $\mathbb{I}$ while leaving behind a set of cardinality $2^{\aleph_0}$.



Begin by removing the middle third of the unit interval, excluding the endpoints, then repeat iteratively on remaining sub-intervals.

$$\text{Iteration } k \text{ leaves } \left(\frac{2}{3}\right)^k \text{ of the original interval.}$$

Cantor's Ternary Set — Part 2

After k steps of removing middle-thirds, there are 2^k subintervals left.

The removal process NEVER removes an endpoint.

You could only do a finite number of removals, but Cantor supposed it could be done an infinite number of times.

Cantor's infinite process leaves all the endpoints in place.

Cantor's Ternary Set — Part 3

Cantor's infinite process scoops out a total length of sub-intervals

$$\lim_{k\to\infty} 1 - (2/3)^k = 1$$

That squeezes out all the remaining subintervals, but leaves behind a countably infinite ($\aleph_0$) set of endpoints. That's not many points.

But Cantor's process also leaves behind all the *limit points* of the endpoints,**

which leaves $2^{\aleph_0}$ points.** That's a lot of points, as many as on the whole unit interval $\mathbb{I}$!

(**Find proofs on Internet, e.g., Wikipedia for "Cantor set.")

*Cantor's ideas were rejected fiercely at first.***

Theologians: Cantor challenges the absolute infinity of God.

Poincaré: Cantor's ideas are a grave disease.

Kronecker: Cantor is a charlatan, renegade, corrupter of youth.

Wittgenstein: Set theory is utter nonsense, laughable, wrong.

BUT Hilbert defended Cantor: "No one shall expel us from the Paradise that Cantor has created."

Russell: Cantor was one of the greatest intellects of the nineteenth century.

Today Cantor's ideas stand in the foundations of mathematics.

(**See Dauben in references.)

Things you can investigate about "Cantor's Paradise"

There exists a continuous function on the unit interval that is constant on the removed middle-thirds, but climbs to 1 on the Cantor set — those are very narrow steps indeed! (Research the "Devil's staircase" or "Cantor function.")

Paradoxes of set theory alerted logicians to the need for axioms. For example, Cantor had shown that for any set, the set of all its subsets has a larger cardinal, but that imples there can't be a *set of all sets*. (Find out about *Russell's Paradox* and the *Zermelo-Fraenkel* axioms.)

In Cantor's diagonal argument, I said you could pick a non-nine digit for each entry in the list on the diagonal. Should I have specified a definite rule for the choice, or is it possible to just assume an infinite number of choices can be made? (Look up the *Axiom of Choice*.)

After you've thought about cardinal numbers, consider Cantor's ordinal numbers: he designated the cardinality of the countable ordinals as $\aleph_1$ and stated his Continuum Hypothesis: $\aleph_1 = 2^{\aleph_0}$. Stillwell (referenced) has a nice treatment.

The continuum is a human artifact.

Mathematics is a Pandora's Box. Wondrous strange things fly out when new axioms are added.

It's a growing human creation. Maybe our present logic and set theory aren't the best that can be done.

Maybe someone in this audience will improve it.

Recommended reading 1

A grand biography: Joseph Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Princeton, 1979.

These popular books provide rich background:

- William Bloch, *The Unimaginable Mathematics of Borges' Library of Babel*, Oxford, 2008.
- Tobias Dantzig and Joseph Mazur, *Number: The Language of Science*, Plume, 2007.
- Apostolos Doxiadis, Christos H. Papadimitriou, et al, *Logicomix: An Epic Search for Truth*, Bloomsbury USA, 2009.
- John Stillwell, *Roads to Infinity: The Mathematics of Truth and Proof*, A. K. Peters, Ltd, 2010. [Pleasantly rigorous.]

Recommended reading 2 (specialized)

- Georg Cantor , *Contributions to the Founding of the Theory of Transfinite Numbers,* trans and intr by Phiip E. B. Jourdain, Dover Books on Mathematics ed 1955.
- Paul R Halmos, *Naive Set Theory,* Undergraduate Texts in Mathematics, Springer 1974.
- Yiannis Moschovakis, *Notes on Set Theory, 2nd ed,* Undergraduate Texts in Mathematics, Springer 2006.
- Charles Petzold , *The Annotated Turing: A Guided Tour Through Alan Turing's Historic Paper on Computability and the Turing Machine,* Wiley 2008.
- Mary Tiles, *The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise,* Dover Books on Mathematics 2004.

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