The Leibniz Catenary Construction:
Geometry vs Analysis in the 17^{th} Century

Mike Raugh
www.mikeraugh.org

JMM: Joint Mathematics Meetings, Atlanta

January 4, 2017

Catenary: Derived from Latin Word for Chain, Catena

Internet

Some History

1638, Galileo discussed the hanging-chain problem.

1690, Jacob Bernoulli published a challenge to solve the problem within 1 year.

1691, Leibniz and *Johann* Bernoulli published the first solutions.

1761, Johann Heinrich Lambert introduced hyperbolic functions and named them:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Leibniz's solution was presented as a classic "Ruler & Compass" construction.

Paradox?

The Construction is not possible because e is transcendental!

And yet it is correct!

It reveals analytical knowledge of the exponential function, and it depicts a hyperbolic cosine.

(70 years before Lambert!)

Leibniz did not publish the derivation of his construction.

(It was communicated in a private letter.)

And so our story begins....

Analytic Formulation of the Catenary

We can express the catenary in terms of a hyperbolic cosine:

$$y = a \cosh \frac{x}{a}.$$

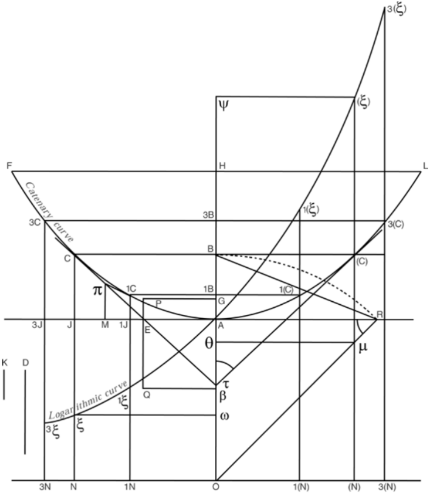
Or in terms of exponentials:

$$y = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}.$$

The curve is bilaterally symmetric about the y-axis,

and the lowest point is at $(0, a)$.

Leibniz's Representation of the Catenary: A Classical Ruler & Compass Construction



The segments D and K are assumed given.

Leibniz uses only their ratio: $\frac{d}{k}$.

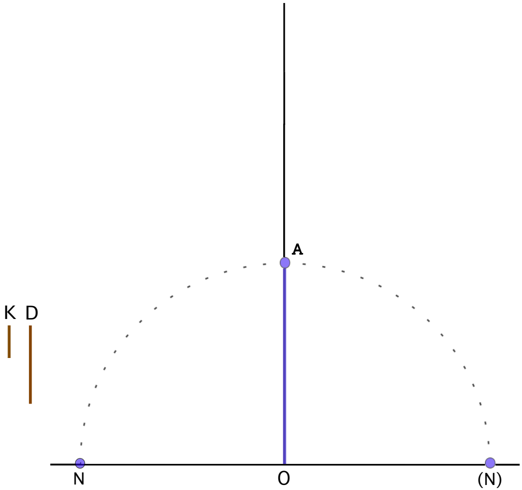
If the ratio is not constructable, then neither is the curve.

But D and K are given, so their ratio could be anything.

This fact can make a fictitious "construction" correct,
(in theory).

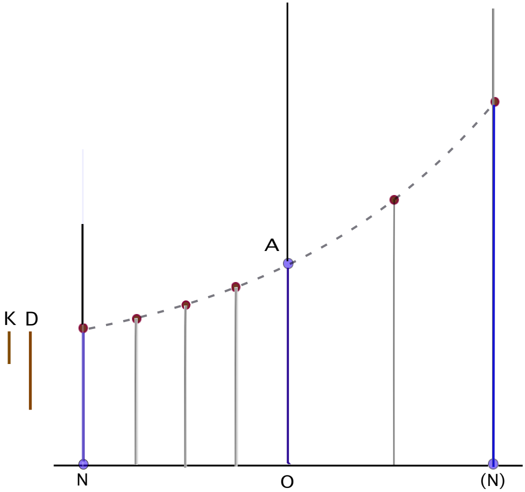
This resolves the paradox for Analysts but not for Geometers.

First Steps of the Construction



Draw: (1) horizontal axis, (2) origin O and vertical axis;
(3) choose $\overline{OA}$ as unit, (4) mark unit lengths on horizontal axis.

Constructing the "Logarithmic Curve"



Ordinates over N & O and O & (N) are in ratio K:D.
Middling ordinates are determined by geometric means.

The "Logarithmic Curve" in Cartesian Coordinates (Represented as an Exponential Curve)

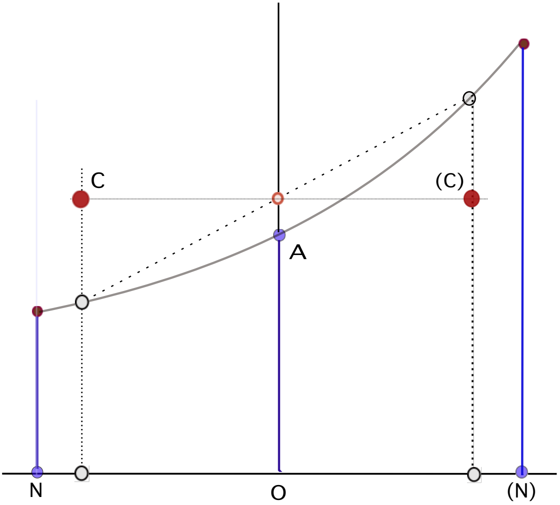
Given two points (x_1, y_1) and (x_2, y_2), get a new one:

$$\left(\frac{x_1 + x_2}{2},\ \sqrt{y_1 y_2}\right)$$

The construction yields dense points on the curve,

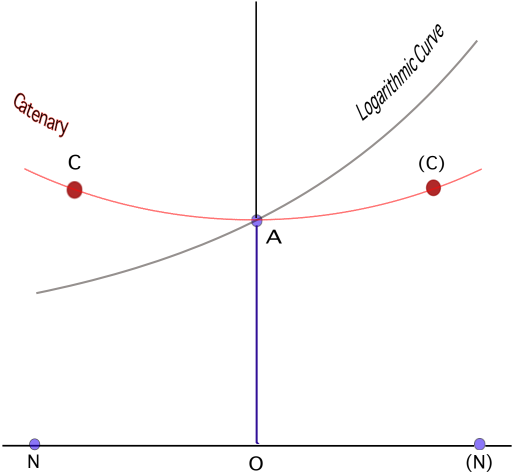
$$y(x) = a\left(\frac{d}{k}\right)^{x/a} \quad (x \text{ constructible})$$

Construction of the "Catenary"



As constructed: $C(x) = \dfrac{r^x + r^{-x}}{2}$, with $a = 1$ and $r = \dfrac{d}{k}$

Leibniz's "Catenary" is Built on an Exponential Curve.



$$z(x) = \frac{a}{2} \cdot \left\{ \left(\frac{d}{k}\right)^{\frac{x}{a}} + \left(\frac{d}{k}\right)^{-\frac{x}{a}} \right\}$$

Leibniz's "Catenary" in Cartesian Coordinates

A true catenary must be of the form:

$$z(x) = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$$

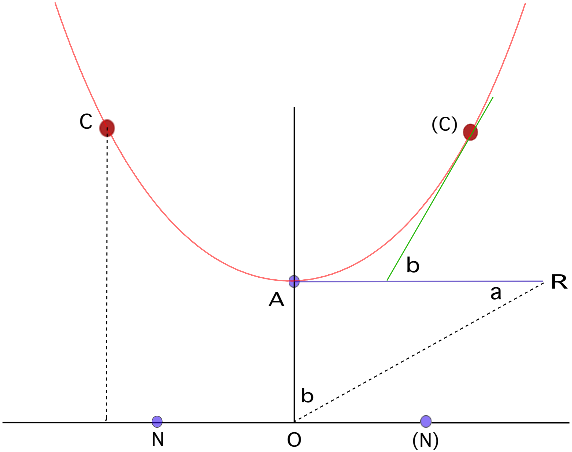
Leibniz needed the ratio $d/k = e$.

In effect, that is what he used, revealed in his figure.

Leibniz's catenary is not constructible, in the sense of Euclid!
But it does correctly characterize the catenary!

Leibniz accepted curves based on analysis vs only those allowed in Cartesian geometry.

Two Examples Requiring a True Catenary: $d/k = e$

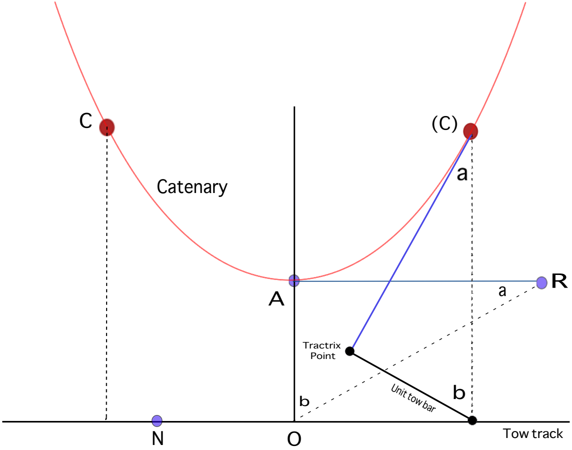


Segment $\overline{\text{AR}}$ is equal in length to arc $\widehat{\text{CA}}$.

Tangent at (C) follows from fact that ∠b is the complement of ∠a.

$(y = \cosh x)$

For Fun: The Tractrix is the Involute of the Catenary.



Rotate the arc-length triangle to trace a tractrix.
(A problem solved by Leibniz later, not in his figure.)

How did Leibniz arrive at his solution?

He explained his derivation in a letter:

To Rudolf Christian von Bodenhausen, August 1691, with attached Latin text, “Analysis problematis catenarii”, in G. W. Leibniz, Sämtliche Schriften und Briefe, series III, volume 5 (2003), p. 143-155

Leibniz does not mention e explicitly. Instead he uses a differential equation for the natural logarithm.

(Thanks to Siegmund Probst)

A Derivation Based on Leibniz Argument, but Simpler:

Leibniz deduced, as did Bernoulli (see Ferguson), that:

$$\frac{dy}{dx} = y' = \frac{s(x)}{a} \quad \Longrightarrow \quad dx = \frac{a\,dy}{\sqrt{y^2 + 2ay}}$$

($s =$ arc length, a constant)

Setting $z = y + a$, Leibniz inferred $s = \sqrt{z^2 - a^2}$

Here I depart from Leibniz to write: $z^2 - s^2 = a^2$,

and let $a = 1$: $(z - s)(z + s) = 1$.

A clue: $(z-s)(z+s)=1.$

The catenary is bilaterally symmetric.

And for $P \cdot Q = 1$,

$\ln P$ and $\ln Q$ are also symmetric about the origin.

This hints at a role for e^x, but $y = e^x$ isn't symmetric.

So symmetrize: $Z = \dfrac{e^x + e^{-x}}{2}$,

and find S such that $S^2 = Z^2 - 1$:

Solution: $S = \dfrac{e^x - e^{-x}}{2}$

Z and S satisfy necessary conditions, but we also need that S = arc length.

Z and S satisfy these equations:

$$(Z-S)(Z+S)=1, \quad \text{and} \quad \frac{dZ}{dx}=S$$

But does S = arc length?

YES, because of these definitions and obvious results:

$$Z=\cosh x \equiv \frac{e^x+e^{-x}}{2}, \; S=\sinh x \equiv \frac{e^x-e^{-x}}{2}$$

$$\frac{dc}{dx}=s, \; \frac{ds}{dx}=c, \quad Z^2-S^2=\cosh^2-\sinh^2 x=1$$

Use the above to validate constructions for the tangent and arc length:

The construction for the tangent implies:

$$Z'(x) = \sqrt{Z^2 - 1}$$

And arc length follows because,

$$S = \int \sqrt{1 + Z'^2}\, dx = \int \cosh x\, dx = \sinh x = \sqrt{Z^2 - 1}$$

The coordinates (x, Z) represent a point C on the catenary.
Integration is over the interval from $(0, 1)$ to (x, Z).

Conclusion

In 1761 Lambert named the “Hyperbolic Cosine”:

$$y = \frac{e^x + e^{-x}}{2}.$$

In 1691 Leibniz had already called it the Catenary!

Leibniz used conventional constructions to exhibit curves, but he relied on analysis as well.

At the time of Leibniz, the Cartesian canon of construction began yielding to the tools of calculus.

Acknowledgments

Invitation and early discussions of the topic:

Adrian Rice, Randolph-Macon College

Communications about Leibniz translations and literature:

Eberhard Knobloch, Berlin-Brandenburg Academy of Science
Siegmund Probst, Leibniz Archive, Göttingen Science Academy

Cautions about modern vs historic ways of thinking:

Robert E. Bradley, Adelphi University

Discussion about the tractrix and tech:

Jorge Balbás, California State University, IPAM

References I

Bernoulli, Johann, *Lectures on the Integral Calculus, Part III*, (Ferguson Translation), pdf on Internet, 2004.

Bos, Henk , *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*, Springer, 2001.

Guicciardini, Niccolo, *Isaac Newton on Mathematical Certainty and Method*, MIT Press, 2011.

Hoffman, Joseph, *Leibniz in Paris 1672 – 1676, His Growth to Mathematical Maturity*, Cambridge University Press, 1974.

Leibniz, Gottfried Wilhelm, *Die Mathematischen Zeitschriftenartikel* (German translation and comments by Hess und Babin), Georg Olms Verlag, 2011. (Also see Ferguson's English translation on Internet.)

Wikipedia and Internet used for dates and graphics.

References II

For a scholarly treatment of methods and notation used by the earliest followers of Leibniz, see,

L'hôpital's analyse des infiniments petits: an annotated translation with source material by Johann Bernoulli, edited by Robert E. Bradley et al, Birkhauser 2015.

For a brisk reprise of Leibniz's explanation of how to use a catenary to determine logarithms, see,

Viktor Blåsjö, *How to Find the Logarithm of Any Number Using Nothing But a Piece of String*, vol 47(2) March 2016, The College Mathematics Journal.

For some of Leibniz's own earliest work from a MS written in 1676 during his years in France, see,

G. W. Leibniz, *quadratura arithmetica circuli, ellipseos et hyperbolae*, Herausgegeben und mit einem Nachwort versehen von Eberhard Knobloch, aus dem Lateinishen übersetzt von Otto Hamborg, Springer Spektrum 2016.

A Monumental Catenary Arch 631 Feet High

The Gateway Arch, St. Louis, Missouri

Internet

Thanks for your attention.

Supplementary notes (and these slides) available at,

www.mikeraugh.org

For questions or comments, please write to Mike at:

Auranteacus@gmail.com