# *The Leibniz Catenary:* *"Let those who don't know the new analysis try their luck!"*

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# *Catenary: Derived from Latin Word for Chain, Catena*

Internet

## *Some History*

1638, Galileo discussed the hanging-chain problem.

1690, Jacob Bernoulli published a challenge to solve the problem within 1 year.

1691, Leibniz and *Johann* Bernoulli published the first solutions.

1761, Johann Heinrich Lambert introduced hyperbolic functions and named them:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

## *Leibniz's solution was presented as a classic "Ruler & Compass" construction.*

Paradox?
The Construction is not possible because $e$ is transcendental!
And yet it is correct!

It reveals analytical knowledge of the exponential function,
and it displays a hyperbolic cosine.
(70 years before Lambert!)

Leibniz did not publish the derivation of his construction.
It was communicated in a private letter.

And so our story begins....

# *Analytic Formulation of the Catenary*

We can express the catenary in terms of a hyperbolic cosine:

$$\frac{y}{a} = \cosh \frac{x}{a} \qquad (a = \text{scaling factor}).$$

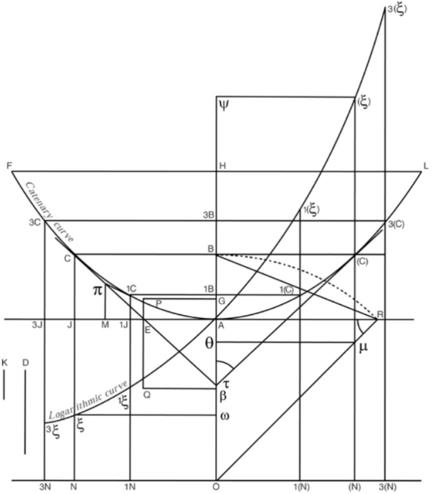
Or in terms of exponentials:

$$y = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}.$$

The curve is bilaterally symmetric about the $y$-axis,

and the lowest point is at $(0, a)$.

# *Leibniz's Representation of the Catenary:*
# *A Classical Ruler & Compass Construction*



## *The segments D and K are assumed given.*

Leibniz uses only their ratio: $\frac{d}{k}$.

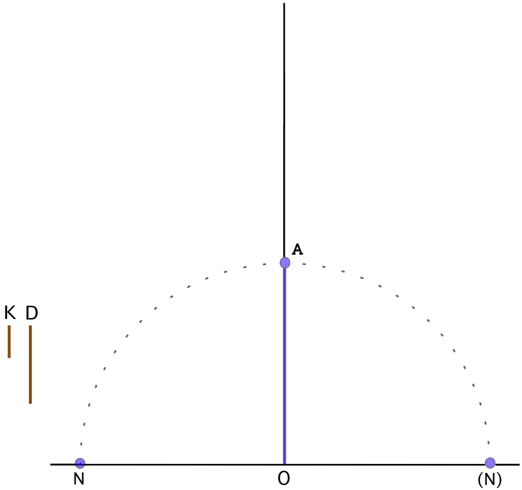
If the ratio is not constructable, then neither is the curve.

But D and K are given, so their ratio could be anything.

This fact can make a fictitious “construction” correct,
(in theory).

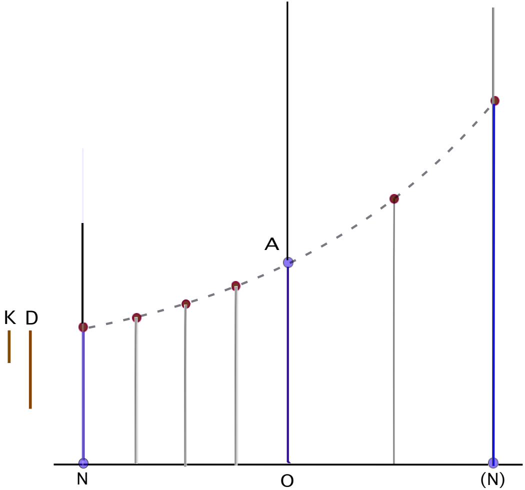
This resolves the paradox for Analysts but not for Geometers.

# *First Steps of the Construction*




Draw: (1) horizontal axis, (2) origin O and vertical axis;
(3) choose $\overline{OA}$ as unit, (4) mark unit lengths on horizontal axis.

# *Constructing the "Logarithmic Curve"*




Ordinates over N & O and O & (N) are in ratio K:D.
Middling ordinates are determined by geometric means.

# *The "Logarithmic Curve" in Cartesian Coordinates (Represented as an Exponential Curve)*

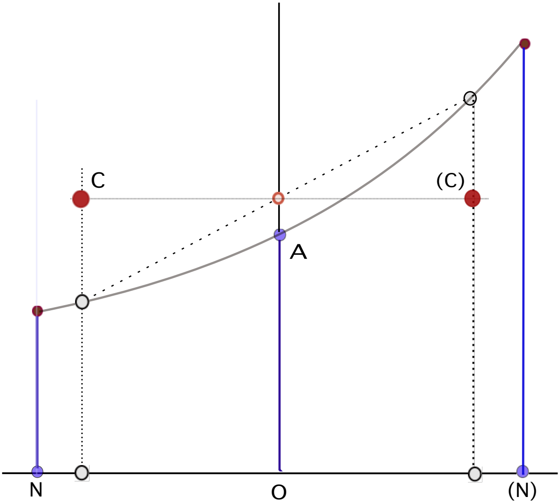
Given two points $(x_1, y_1)$ and $(x_2, y_2)$, get a new one:

$$\left(\frac{x_1 + x_2}{2},\ \sqrt{y_1 y_2}\right)$$

The construction yields dense points on the curve,

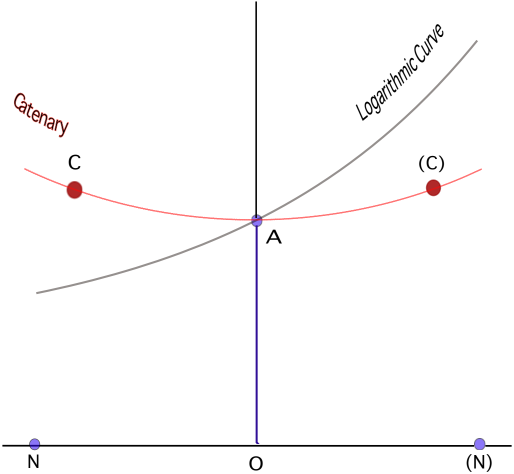
$$y(x) = a\left(\frac{d}{k}\right)^{x/a} \quad (x \text{ a binary number})$$

# *Construction of the "Catenary"*




As constructed: $C(x) = \dfrac{r^x + r^{-x}}{2}$, with $a = 1$ and $r = \dfrac{d}{k}$

# *Leibniz's "Catenary" is Built on an Exponential Curve.*




$$z(x) = \frac{a}{2} \cdot \left\{ \left(\frac{d}{k}\right)^{\frac{x}{a}} + \left(\frac{d}{k}\right)^{-\frac{x}{a}} \right\}$$

# *Is Leibniz's "catenary" truly a catenary?*

A true catenary must be of the form:

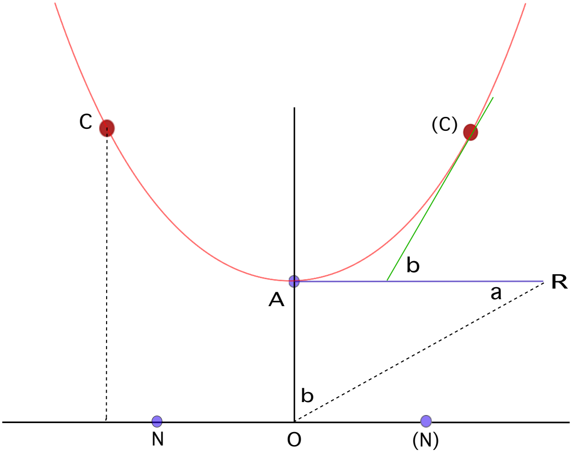
$$z(x) = a \cdot \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$$

Leibniz needed the ratio $d/k = e$.

In effect, he used that — as revealed in his figure.

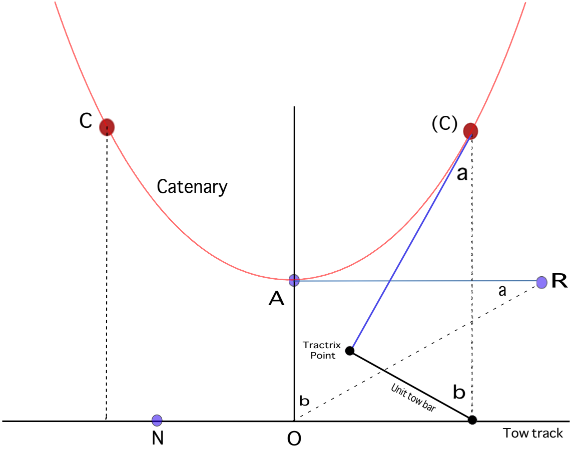
So, as we shall see, it is a true catenary.

# *Two Examples Requiring a True Catenary:* $d/k = e$




Segment $\overline{\text{AR}}$ is equal in length to arc $\widehat{\text{CA}}$.
Tangent at (C) follows from fact that $\angle$b is the complement of $\angle$a.
$(y = \cosh x)$

# *For Fun: The Tractrix is the Involute of the Catenary.*




Rotate the arc-length triangle to trace a tractrix.
(A problem solved by Leibniz later, not in his figure.)

# *How did Leibniz arrive at his solution?*

He explained his derivation in a letter:

To Rudolf Christian Von Bodenhausen, August 1691, with attached Latin text, “Analysis problematis catenarii”, in G. W. Leibniz, Sämtliche Schriften und Briefe, series III, volume 5 (2003), p. 143-155

(Thanks to Siegmund Probst)

# *The Derivation Part 1:*

Leibniz deduced, as did Bernoulli (see Ferguson), that:

$$\frac{dy}{dx} = y' = \frac{s(x)}{a} \qquad \Longrightarrow \qquad dx = \frac{a\,dy}{\sqrt{y^2 + 2ay}}$$

($s =$ arc length, $a =$ scaling factor)

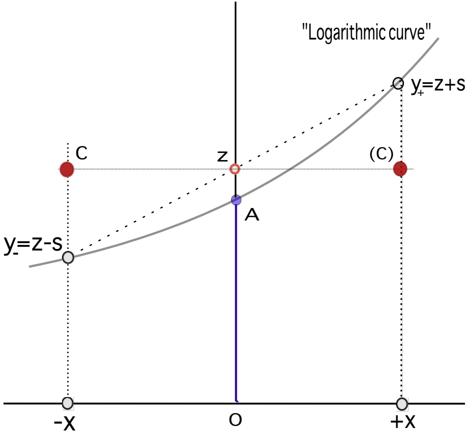
Setting $z = y + a,$ Leibniz inferred $s = \sqrt{z^2 - a^2}$

Or, $z^2 - s^2 = a^2$.

From this, the leading clue:

(We can use $a = 1$): $(z - s)(z + s) = 1$.

# *A solution is suggested by this graph:*




$$(z-s)(z+s)=1$$

Let: $-x=\ln(z-s),\ +x=\ln(z+s)$.

Then: $z=\dfrac{y_-+y_+}{2}$ (Why not $\log_b, b\neq e$?)

# *The Derivation Part 2*

$(z-s)(z+s)=1$ suggests: $\ln(z-s)=-\ln(z+s)$.

Leibniz approach: $\omega(x)=z-s$ and use $d(\ln\omega)$:

$$\frac{d\omega}{\omega}=\frac{dz-ds}{z-s}$$

But $z\,dz=s\,ds$ and $dz=s\,dx \quad \Longrightarrow$

$$\frac{d\omega}{\omega}=\frac{dz-z\,dx}{z-dz/dx}=-dx \quad \Longrightarrow$$

$-x=\ln(z-s)$, similarly $+x=\ln(z+s)$

[ICs $z(0)=1, z'(0)=0 \implies$ Integration constant $=0$.]

# The Derivation Part 2b: Bring back "a"

Leibniz (simplified): $\dfrac{(z-s)}{a}\dfrac{(z+s)}{a}=1;$ let: $\omega=\dfrac{(z-s)}{a}$.

$$\frac{d\omega}{\omega}=\frac{dz-ds}{z-s}=-\frac{dx}{a}$$

So,

$$-\frac{x}{a}=\ln\frac{(z-s)}{a}, \quad \text{similarly} \quad +\frac{x}{a}=\ln\frac{(z+s)}{a}$$

$$\frac{z}{a}=\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2}=\cosh\frac{x}{a}$$

[ICs $z(0)=a, z'(0)=0 \implies$ Integration constant $=0$.]

*"Let those who don't know the new analysis try their luck!"*

To Bodenhausen Leibniz also wrote that

he left out one specification for his construction:

!!! D and K were in ratio 1 to 2.7182818 !!!

Why this ostentatious *approximation*?

He didn't name it, need it or explain it,
and it was too precise to use.

He had already built-in the exact value for $e$ using $\frac{d\omega}{\omega}$.

## *A Sample of Leibniz's Technique: Differentials I*

Prove: $\dfrac{dy}{dx} = s \quad \Longrightarrow \quad dx = \dfrac{dy}{\sqrt{y^2 + 2y}}$

Use: $(ds)^2 = (dx)^2 + (dy)^2$

Let $dx$ be constant, differentiate:

$ddy = ds\,dx$ and $ds\,dds = dy\,ddy$

Combine (and "anti-differentiate"):

$$dds = dy\,dx \quad \Longrightarrow \quad ds = y\,dx + cdx \quad \Longrightarrow$$

$$(dx)^2 + (dy)^2 = (y^2 + 2y + c)(dx)^2$$

# *A Quick Sample of Technique: Differentials II*

$$(dx)^2 + (dy)^2 = (y^2 + 2y + c)(dx)^2 \qquad \Longrightarrow$$

[Use initial conditions, $\quad y(0) = y'(0) = 0$]

$$(dy)^2 = (y^2 + 2y)(dx)^2$$

$$\frac{dy}{dx} = \sqrt{y^2 + 2y} \qquad \Longrightarrow \qquad dx = \frac{dy}{\sqrt{y^2 + 2y}}$$

QED

# *Conclusion*

In 1761 Lambert named the "Hyperbolic Cosine":

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

In 1691 Leibniz had already called it the Catenary!

At the time of Leibniz, the Cartesian canon of construction began yielding to the methods of calculus.

Leibniz used conventional constructions to exhibit curves, but he relied on analysis as well.

That comment about 2.7182818:
Why??

# *Acknowledgments*

Communications about Leibniz translations and literature:

Eberhard Knobloch, Berlin-Brandenburg Academy of Science

Siegmund Probst, Leibniz Archive, Göttingen Science Academy

Discussions about the use of differentials in early calculus:

Robert Bradley, Adelphi University

Discussion about the tractrix:

Jorge Balbás, California State University, IPAM

# *References I*

Bernoulli, Johann, *Lectures on the Integral Calculus, Part III*, (Ferguson Translation), pdf on Internet, 2004.

Bos, Henk , *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*, Springer, 2001.

Guicciardini, Niccolo, *Isaac Newton on Mathematical Certainty and Method*, MIT Press, 2011.

Hofmann, Joseph, *Leibniz in Paris 1672 – 1676, His Growth to Mathematical Maturity*, Cambridge University Press, 1974.

Leibniz, Gottfried Wilhelm, *Die Mathematischen Zeitschriftenartikel* (German translation and comments by Hess und Babin), Georg Olms Verlag, 2011. (Also see Ferguson's English translation on Internet.)

Wikipedia and Internet used for dates and graphics.

# *References II*

For a scholarly treatment of methods and notation used by the earliest followers of Leibniz, see,

*L'hôpital's analyse des infiniments petits: an annotated translation with source material by Johann Bernoulli*, edited by Robert E. Bradley et al, Birkhauser 2015, (pp 62-65 for use of $dx$ as free variable).

For a brisk reprise of Leibniz's explanation of how to use a catenary to determine logarithms, see,

Viktor Blåsjö, *How to Find the Logarithm of Any Number Using Nothing But a Piece of String*, vol 47(2) March 2016, The College Mathematics Journal.

For some of Leibniz's own earliest work from a MS written in 1676 during his years in France, see,

G. W. Leibniz, *quadratura arithmetica circuli, ellipseos et hyperbolae*, Herausgegeben und mit einem Nachwort versehen von Eberhard Knobloch, aus dem Lateinishen übersetzt von Otto Hamborg, Springer Spektrum 2016.

# A Monumental Catenary Arch 631 Feet High

The Gateway Arch, St. Louis, Missouri

Internet

# *Thanks for your attention.*

Supplementary notes (and these slides) available at,

www.mikeraugh.org

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