The Bernoulli Summation Formula: A Pretty Natural Derivation

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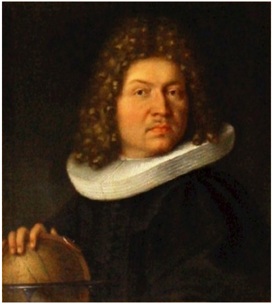
Jakob Bernoulli's Summation Formula



(1655–1705, Wikimedia Commons)

Bernoulli specified the form of the coefficients a_j in a general polynomial formula for summing powers of the integers:

$$\sum_{j=1}^{n} j^k = a_0 n^{k+1} + a_1 n^k + a_2 n^{k-1} + \cdots + a_n n$$

We will "rediscover" Bernoulli's formulation.

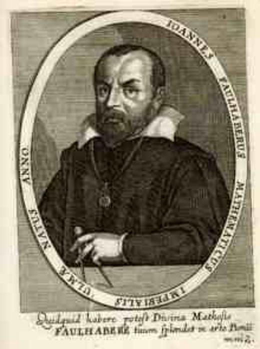
Bernoulli credited Johann Faulhaber, "The Calculator of Ulm," for discovering specific formulas for $k = 1, \cdots, 17$.

Jakob Bernoulli's formula was for general k.

Certain constants in Bernoulli's formula are known as *Bernoulli numbers*, named by Abraham De Moivre (& Leonhard Euler).

Our derivation follows easily from *inspecting a table of formulas*.

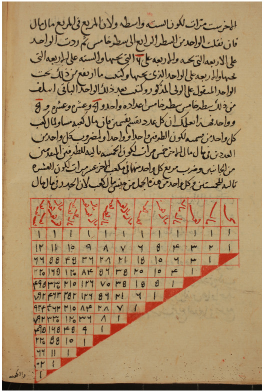
Johann Faulhaber



(1580–1635, Wikimedia Commons)

Faulhaber, surveyor and instrument maker, collaborated with Kepler, supervised publication of Briggs's logarithms in Germany, later famed for summation formulas for powers of integers. Donald Knuth broke secret code in which Faulhaber gave formulas correct up to $k = 23$.

We will need the Binomial Coefficients.



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Predating "Pascal's Triangle", a MS copy dated 1324 by nineteen-year-old Islamic mathematician. (Article in AMS Notices of Dec 2013) Earliest commentary however from Hindi mathematician Halayuda from 10th-century.

Pascal's Triangle and Binomial Coefficients

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(Wikimedia Commons)

The "n^{th}" row gives coefficients for $(a+b)^n$:

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j; \qquad \binom{n}{j} = \frac{n!}{j!(n-j)!}$$

The symbol is due to Ettingshausen (1796–1878).

Three Things to Keep in Mind

1) If $p(x) = b_0 x^k + b_1 x^{k-1} + \cdots + b_k$ is a polynomial with constant coefficients, and $p(x) = 0$ for infintely many distinct values of x, then $p(x) = 0$ for *all real values* of x. (Why?) So, if $g(x)$ is another polynomial such that $f(x) = g(x)$ for an infinite number of integers, then equality holds for $x \in \mathbb{R}$, i.e., all real values of x. (Why?)

2) Suppose $S_k(n)$ is a polynomial such that $S_k(n) = \sum_{j=1}^{n} j^k$ for a fixed non-negative integer k and all positive integers n, then by (1), $S_k(x) - S_k(x-1) = x^k$, for all $x \in \mathbb{R}$. (Why?)

3) For any function F defined for all non-negative integers, $\sum_{j=1}^{n} [F(j) - F(j-1)] = F(n) - F(0)$. The sum is said to *telescope*, i.e., all but the first and last terms cancel out. (Why?)

First Summation Method (A More Efficient One will Follow)

Suppose we already have a formula, say,

$$\sum_{j=1}^{n} j = \frac{1}{2}n^2 + \frac{1}{2}n$$

Then sum for $j = 1$ to n the identity $j^3 - (j-1)^3 = 3j^2 - 3j + 1$:

Telecoping sum: $\quad n^3 - 0^3 = 3\sum_{j=1}^{n} j^2 - 3\sum_{j=1}^{n} j + n$

Insert the above for $\sum_{j=1}^{n} j$ to get:

$$\sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

First Method Repeated

Now we have,

$$\sum_{j=1}^{n} j = \frac{1}{2}n^2 + \frac{1}{2}n \quad \text{and} \quad \sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

So now sum the identity $j^4 - (j-1)^4 = 4j^3 - 6j^2 + 4j - 1$.

Get: $n^4 - 0^4 = 4\sum_{j=1}^{n} j^3 - 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j - n$

Inserting the above, find:

$$\sum_{j=1}^{n} j^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n$$

Continuing the process:

$$\sum n^0 = \tfrac{1}{1} n$$
$$\sum n^1 = \tfrac{1}{2} n^2 + \tfrac{1}{2} n$$
$$\sum n^2 = \tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n$$
$$\sum n^3 = \tfrac{1}{4} n^4 + \tfrac{1}{2} n^3 + \tfrac{1}{4} n^2 + 0 n$$
$$\sum n^4 = \tfrac{1}{5} n^5 + \tfrac{1}{2} n^4 + \tfrac{1}{3} n^3 + 0 n^2 - \tfrac{1}{30} n$$
$$\sum n^5 = \tfrac{1}{6} n^6 + \tfrac{1}{2} n^5 + \tfrac{5}{12} n^4 + 0 n^3 - \tfrac{1}{12} n^2 + 0 n$$
$$\sum n^6 = \tfrac{1}{7} n^7 + \tfrac{1}{2} n^6 + \tfrac{1}{2} n^5 + 0 n^4 - \tfrac{1}{6} n^3 + 0 n^2 + \tfrac{1}{42} n$$
$$\sum n^7 = \tfrac{1}{8} n^8 + \tfrac{1}{2} n^7 + \tfrac{7}{12} n^6 + 0 n^5 - \tfrac{7}{24} n^4 + 0 n^3 + \tfrac{1}{12} n^2 + 0 n$$
$$\sum n^8 = \tfrac{1}{9} n^9 + \tfrac{1}{2} n^8 + \tfrac{2}{3} n^7 + 0 n^6 - \tfrac{7}{15} n^5 + 0 n^4 + \tfrac{2}{9} n^3 + 0 n^2 - \tfrac{1}{30} n$$
$$\sum n^9 = \tfrac{1}{10} n^{10} + \tfrac{1}{2} n^9 + \tfrac{3}{4} n^8 + 0 n^7 - \tfrac{7}{10} n^6 + 0 n^5 + \tfrac{1}{2} n^4 + 0 n^3 - \tfrac{7}{20} n^2 + 0 n$$

$$-\frac{3}{20}$$

$$\sum n^{10} = \tfrac{1}{11} n^{11} + \tfrac{1}{2} n^{10} + \tfrac{5}{6} n^9 + 0 n^8 - 1 n^7 + 0 n^6 + 1 n^5 + 0 n^4 - \tfrac{1}{2} n^3 + 0 n^2 + \tfrac{5}{66} n$$

(Notes of a Twentieth Century Teenager)

$\sum n^k$ is used as shorthand for $\sum_{j=1}^{n} j^k$.

There is a progression in coefficients from one line to the next.

Can you see it? (Try integration!)

Invalid integration, but try it!

Pick two consecutive summation formulas from the table.

Pretend that n is a continuous variable, then integrate the first equation and compare with the second equation:

$$\sum n^3 = \tfrac{1}{4} n^4 + \tfrac{1}{2} n^3 + \tfrac{1}{4} n^2 + 0 n$$

$$\sum n^4 = \tfrac{1}{5} n^5 + \tfrac{1}{2} n^4 + \tfrac{1}{3} n^3 + 0 n^2 - \tfrac{1}{30} n$$

Let's see what we get.

Here we go:

$$\Sigma n^3 = 1/4\, n^4 + 1/2\, n^3 + 1/4\, n^2 + 0 n$$

$$\sum \int n^3 = \frac{1}{4} \int n^4 + \frac{1}{2} \int n^3 + \frac{1}{4} \int n^2 + 0 \int n$$

$$\sum \frac{n^4}{4} = \frac{1}{4}\frac{n^5}{5} + \frac{1}{2}\frac{n^4}{4} + \frac{1}{4}\frac{n^3}{3} + 0\frac{n^2}{2}$$

The result is missing (only) the linear term:

$$\sum n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + 0n^2 \qquad -\frac{1}{30}n$$

$$\Sigma n^4 = 1/5\, n^5 + 1/2\, n^4 + 1/3\, n^3 + 0 n^2 - 1/30\, n$$

Why the near miss?

Consider the previous summation formula as a polynomial,

DEFINED AS: $$S_3(n) \equiv \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \sum_{j=1}^{n} j^3$$

Then the polynomial $S_3(n)$ satisfies the following equation for infinitely many integer values of n:

$$S_3(n) - S_3(n-1) = n^3, \quad n \in \mathbb{N}_{>1}$$

Therefore it is satisfied for all real values of x:

$$S_3(x) - S_3(x-1) = x^3 \quad x \in \mathbb{R}$$

This polynomial equation can be integrated!

Now integration is valid, and we can sum the result.

$$\int_0^j S_3(x)\,dx - \int_0^j S_3(x-1)\,dx = \int_0^j x^3\,dx = \frac{1}{4}j^4$$

Rewrite the second integral:

$\int_0^j S_3(x-1)\,dx = \int_{-1}^{j-1} S_3(x)\,dx = \int_0^{j-1} S_3(x)\,dx + \int_{-1}^0 S_3(x)\,dx$

So for some constant C_3,

$$\left[\int_0^j S_3(x)\,dx - \int_0^{j-1} S_3(x)\,dx\right] + C_3 = \frac{1}{4}j^4$$

Sum the equation(s) for $j = 1$ to n to find the missing term:

$$\int_0^n S_3(x)\,dx + C_3 n = \frac{1}{4}\sum_{j=1}^{n} j^4$$

This efficient technique works in general.

If $S_k(n) = \sum_{j=1}^{n} j^k$, then

$$\int_0^n S_k(x)\,dx + C_k n = \frac{1}{k+1}\sum_{j=1}^{n} j^{k+1}$$

Therefore,

$$\sum_{j=1}^{n} j^{k+1} = S_{k+1}(n) = (k+1)\int_0^n S_k(x)\,dx + (k+1)C_k n$$

By convention, we define the Bernoulli Numbers:

$$B_{k+1} = (k+1)C_k.$$

Again, in terms of Bernoulli Numbers:

Given $S_k(n) = \sum_{j=1}^{n} j^k$, integrate to get,

$$\sum_{j=1}^{n} j^{k+1} = S_{k+1}(n) = (k+1)\int_0^n S_k(x)\,dx + B_{k+1}n$$

The integration is easy, but we need to know the Bernoulli numbers,

$$B_0, B_1, B_2, B_3, \cdots.$$

We'll get back to this, so note for later that the integration above shows that for $k > 0$, $S_k(n)$ has no constant term, therefore,

$$S_k(x) - S_k(x-1) = x^k \implies S_k(-1) = 0, \text{ for } k > 0.$$

Repetition leads to the Bernoulli Summation Formula

Instead of computing the Bernoulli numbers, leave their symbols in place. Iterate to get Bernoulli's Summation Formula:

$$\sum_{j=1}^{n} j^k = S_k(n) = \sum_{j=0}^{k} \frac{k!}{j!} B_j \frac{n^{k+1-j}}{(k+1-j)!} =$$

$$\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j}$$

Symbolically,

$$\sum_{j=1}^{n} j^k = \frac{1}{k+1} \left\{ (n+B)^{k+1} - B^{k+1} \right\}; \quad B^m \to B_m$$

Use Arithmetic to Compute Bernoulli Numbers

They're tedious to calculate, and you can find them in tables.

But let's do this!

Begin with $S_0(n) = \sum_{j=1}^{n} j^0 = n$. Integrate:

$$S_1(n) = 1 \cdot \int_0^n S_0(x)\,dx + B_1 n = \frac{1}{2}n^2 + B_1 n$$

Setting $n = -1$ and using $S_1(-1) = 0$ yields,

$$0 = \frac{1}{2} - B_1 \implies B_1 = \frac{1}{2} \implies \sum_{j=1}^{n} j^1 = \frac{1}{2}n^2 + \frac{1}{2}n$$

Let's do it again!

Start with,

$$\sum_{j=1}^{n} j^1 = S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

Integrate to get,

$$S_2(n) = 2\int_0^n S_1(x)\,dx + B_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + B_2 n$$

Setting $n = -1$ again and using $S_2(-1) = 0$ yields,

$$0 = -\frac{1}{3} + \frac{1}{2} - B_2 \implies B_2 = \frac{1}{6} \implies \sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Doing this repeatedly (starting with $\sum n^0 = n$) will give you the table of 11 summation formulas shown earlier.

The First Eleven Bernoulli Numbers

They are the coefficients of n in the earlier table for sums:

$$B_0 = 1$$

$$B_1 = \frac{1}{2}, \quad B_3 = B_5 = B_7 = B_9, \cdots = 0$$

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, \cdots$$

For odd $j > 2$, $B_j = 0$; for even j, the signs alternate. (Proved in Afterwords)

Tables often give $B_1 = -1/2$, in which case the index of summation runs up to $n - 1$ (instead of n):

$$\sum_{j=1}^{n-1} j^k = \frac{1}{k+1}\left\{(n+B)^{k+1} - B^{k+1}\right\};\ B^m \to B_m,\ B_1 = -1/2$$

See further theory in Afternotes for self-study.

After the formal conclusion of this lecture you are guided through proofs of the following:

Every Bernoulli Summation Formula for even $2n > 1$ is factorable by $n(n+1)(2n+1)$, and for every odd $2n+1$ is factorable by $n^2(n+1)$.

The Bernoulli Numbers $B_{2k+1} = 0$ for $2k+1 > 1$, and the B_{2k}'s are non-zero and alternate in sign for $2k > 0$. Proof of the latter is by inspection of Euler's amazing formula for $\zeta(2k)$.

The Bernoulli Summation Formula is derived using the power series expansion for e^x, demonstrating an analytic reason for defining $B_1 = -1/2$.

Euler's and Riemann's zeta functions are discussed in brief, without proofs. Some literature is referenced.

References 1

Bernoulli Numbers and the Bernoulli Summation Formula are classical topics with innumerable references. For a good start see Wikipedia at:

`http://en.wikipedia.org/wiki/Bernoulli_number`

`http://en.wikipedia.org/wiki/Faulhaber%27s_formula`

For a swift though magical (unmotivated) approach akin to the integration in this presentation, see also,

Apostol, Tom, *Calculus, Vol. 1: One-Variable Calculus, with an Introduction to Linear Algebra, 2nd edition*, John Wiley & Sons, Prob. 35 on p225.

(More references follow in the Afternotes.)

Acknowledgments

The graphic of Pascal's Triangle with Arabic script was provided through the kindness of Dr. Ferruh Özpilavcı:

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Thanks for conversations leading to this presentation:

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Kevin Windsor

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Thanks for viewing!

Afternotes for Self-study Online

The theory of this section is sketched with details of proofs left for the reader, or, in the case of the Euler and Riemann zeta functions, simply alluded to. Further study will require going to the literature.

All the material is classical. Some references are given; others abound online and in texts.

Factors of the Bernoulli Summation Formulas

Here I use facts about the Bernoulli summation formulas, $S_k(n)$, developed in the lecture, e.g., that all Bernoulli summation formulas are factorable by n. In this section we assume $k > 0$.

That each Bernoulli Summation Formula is evenly divisible by $n(n+1)$ follows from the fact that $S_k(-1) = 0$.

Now I show that $S_{2k}(-1/2) = 0$, from which divisibility by $n(n+1)(2n+1)$ follows: $S_{2k}(x) - S_{2k}(x-1) = x^{2k}$ implies that $\sum_{j=0}^{n-1}[S_{2k}(-j) - S_{2k}(-j-1)] = -S_{2k}(-n) = \sum_{j=1}^{n-1} j^{2k}$. So $-S_{2k}(-n) = S_{2k}(n-1)$. Setting $n = 1/2$, the result follows.

You will see on the next slide that for all odd $2k+1 > 1$, the linear coefficient for $S_{2k+1} = 0$. It follows that all such S_{2k+1} are evenly divisible by $n^2(n+1)$.

Vanishing of B_{2k+1} *for* $2k+1>1$

From the previous slide, $-S_{2k}(-n) = S_{2k}(n-1)$. Since the right-hand side is the polynomial for $\sum_{j=1}^{n-1} j^{2k}$:

$$S_{2k}(n) + S_{2k}(-n) = n^{2k}$$

So, from the symbolic form of the summation formula (lecture):

$$\frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}\left\{1+(-1)^{2k+1-j}\right\}B_j n^{2k+1-j} = n^{2k}$$

The braces vanish for even j and $B_1 = 1/2$, so for all $k>1$,

$\sum_{j=1}^{k-1}\binom{2k+1}{2j+1}B_{2j+1}n^{2k-2j} = 0$. So, $B_3 = B_5 = B_7 = \cdots = 0$

Introducing Euler's Zeta Function

Instead of summing powers of the integers, we can sum powers of their reciprocals taking the upper limit to be ∞:

$$\zeta(k) \equiv \sum_{j=1}^{\infty} \frac{1}{j^k}, \quad k \in \mathbb{N}_{>1}$$

This is Euler's zeta function. Euler found the amazing formula,

$$\zeta(2k) = \sum_{j=1}^{\infty} \frac{1}{j^{2k}} = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2 \cdot (2n)!}$$

So the even-numbered Bernoulli numbers have to alternate in sign. Euler connected his $\zeta(k)$ to the prime numbers by noting,

$$\zeta(k) = \prod_{j=1}^{\infty} \frac{1}{1 - \frac{1}{p_j^k}}, \quad p_j = 2, 3, 5, 7, 11, \ldots$$

Riemann's Zeta Function

With phenomenal insight, Bernhard Riemann recognized that Euler's zeta-function can be extended into the complex plane, giving us the Riemann zeta function:

$$\zeta(s) \equiv \sum_{j=1}^{\infty} \frac{1}{j^s}, \quad s \in \mathbb{C}_{\neq 1}$$

His astonishing conjecture, the Riemann Hypothesis, is that $\zeta(s)$ vanishes only on the vertical line $\mathrm{Re}\, s = 1/2$, except for the *trivial zeroes* on the negative real axis:

$$\zeta(-k) = \frac{B_{k+1}}{k+1}, \quad k \in \mathbb{N}_{>0}$$

So $\zeta(-2k) = 0$, $k \in \mathbb{N}_{>0}$. Again the Bernoulli numbers arise intriguingly, with distinct behaviors for odd and even integers.

Bernhard Riemann

(1826–1866, Internet)

Riemann Integral, Surfaces, Hypothesis, ...

There is an analytic reason for defining $B_1 = -1/2$

Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we can write:

$$\sum_{j=0}^{n-1} e^{jx} = n + \sum_{k=1}^{\infty} \left(\sum_{j=1}^{n-1} j^k \right) \frac{x^k}{k!} \quad (n > 1)$$

The left-hand side is a geometric progression, summed as,

$$\frac{e^{nx} - 1}{e^x - 1} = \frac{e^{nx} - 1}{x} \cdot \frac{x}{e^x - 1}$$

Expand the second fraction on the right as a power series:

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}, \; \{b_j : j = 0, \ldots, \infty\} \text{ TBD}$$

Analysis 1

Then expand the first fraction on the right,

$$\frac{e^{nx}-1}{e^x-1} = \frac{e^{nx}-1}{x} \cdot \sum_{j=0}^{\infty} b_j \frac{x^j}{j!} =$$

$$\sum_{j=0}^{\infty} \frac{n^{j+1}}{(j+1)!} x^j \cdot \sum_{j=0}^{\infty} b_j \frac{x^j}{j!} =$$

$$\frac{b_0}{0!} \cdot \frac{n}{1!} x^0 + \left(\frac{b_0}{0!} \cdot \frac{n^2}{2!} + \frac{b_1}{1!} \cdot \frac{n}{1!}\right) x^1 + \cdots =$$

$$\sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \frac{b_i}{i!(k+1-i)!} n^{k+1-i} \right) x^k =$$

Analysis 2

Remembering that the coefficient of $x^k/k!$ for $k \geq 1$ was,

$$\left(\sum_{j=0}^{n-1} j^k \right) \frac{x^k}{k!}$$

We can now compare it with,

$$\left(\sum_{i=0}^{k} \frac{b_i}{i!(k+1-i)!} n^{k+1-i} \right) x^k = \left(\frac{1}{(k+1)} \sum_{i=0}^{k} \binom{k+1}{i} b_i n^{k+1-i} \right) \frac{x^k}{k!}$$

Therefore,

$$\sum_{j=0}^{n-1} j^k = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} b_i n^{k+1-i}$$

An Analytic Reason for Defining $B_1 = -1/2$

We have derived two nearly identical summation formulas:

$$\sum_{j=0}^{n-1} j^k = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} b_i n^{k+1-i}$$

$$\sum_{j=0}^{n} j^k = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i n^{k+1-i}, \ (B_1 = \frac{1}{2})$$

Subtract n from the second equation and compare with the first to conclude that $b_1 = -1/2$ and $b_i = B_i$ for all $i \neq 1$.

The Generating Function for the Bernoulli Numbers

So, with the definition $B_1 = -1/2$, we can write:

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j$$

The expression on the left-hand side of the equation is known as the *generating function* for the Bernoulli numbers.

Applications of Bernoulli numbers and polynomials are widespread.

References follow.

References 2

For Bernoulli numbers in number theory, see the extensive treatment in Henri Cohen's *Number Theory, Vol II: Analytic and Modern Tools,* Springer, Ch. 9, "Bernoulli Polynomials and the Gamma Function."

The *Euler-Maclaurin Summation Formula*, discussed by Cohen, uses Bernoulli numbers and yields the Bernoulli Summation Formula as a trivial case.

An excellent treatment of Euler-Maclaurin, discovered first by Euler, is given by Konrad Knopp in *Theory and Application of Infinite Series* (Dover) — a wonderful inexpensive book.

Euler found Bernoulli numbers in various series expansions and exploited their fortuitous properties to obtain his Summation Formula. See V. S. Varadarajan "Euler and His Work on Infinite Series," *Bulletin of the AMS*, Vol 44, No 4, Oct 2007 and *Euler Through Time*, AMS 2000, Chap.4.

References 3

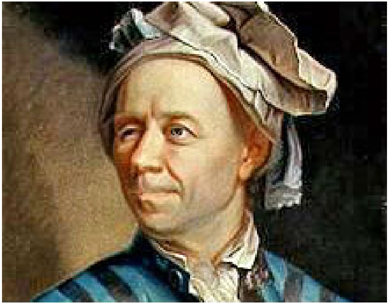
A fine treatment of Euler's formula for $\zeta(2k)$ is given in Richard Courant and Fritz John's *Introduction to Calculus and Analysis, Vol 1*, Interscience Publishers, 1965. (See Appendix II.1, "Bernoulli Polynomials and their Applications." Their "Bernoulli polynomials" are slightly different from the summation polynomials of the lecture.)

Euler, following De Moivre, referred to the constants in his formula for $\zeta(2k)$ as Bernoulli numbers. For an additional treatment leading to Euler's formula see Graham, Knuth and Patashnik's *Concrete Mathematics: A Foundation for Computer Science*, 2^{nd} ed, Addison Wesley 1998. See Chap. 6.5, "Bernoulli Numbers," Eq. 6.89.

Dunham provides one of the approaches Euler made to summing specific instances of the series above in his *Euler: The Master of Us All*, MAA 1999. (See Chapter 3, "Euler and Infinite Series."

For discussions about Euler's methods of analysis, particularly summation, see Ranjan Roy, *Sources in the Development of Mathematics — Series and Products from the Fifteenth to the Twenty-first Century*, Cambridge 2011, various pages.

Leonhard Euler



(Euler portrait by Handmann 1753, Wikimedia Commons)

"Read Euler, Read Euler. He is the master of us all." (Laplace)