Some early implicit and actual appearances of the number commonly known as *e*

International Virtual Symposium in honor of the "Eulerian" number e = 2.71828....

February 7, 18:28 Amsterdam time

Siegmund Probst, Mike Raugh

Additional slides on Bürgi and Kepler from suggestions by Tilman Sauer and Eva Kaufholz-Soldat

e “in disguise”

$$\ln 10 = 2{,}302585092994045684017\ldots = \frac{1}{\log_{10} e}$$

$$\frac{1}{\ln 10} = 0.43429448190325182765\ldots = \log_{10} e$$

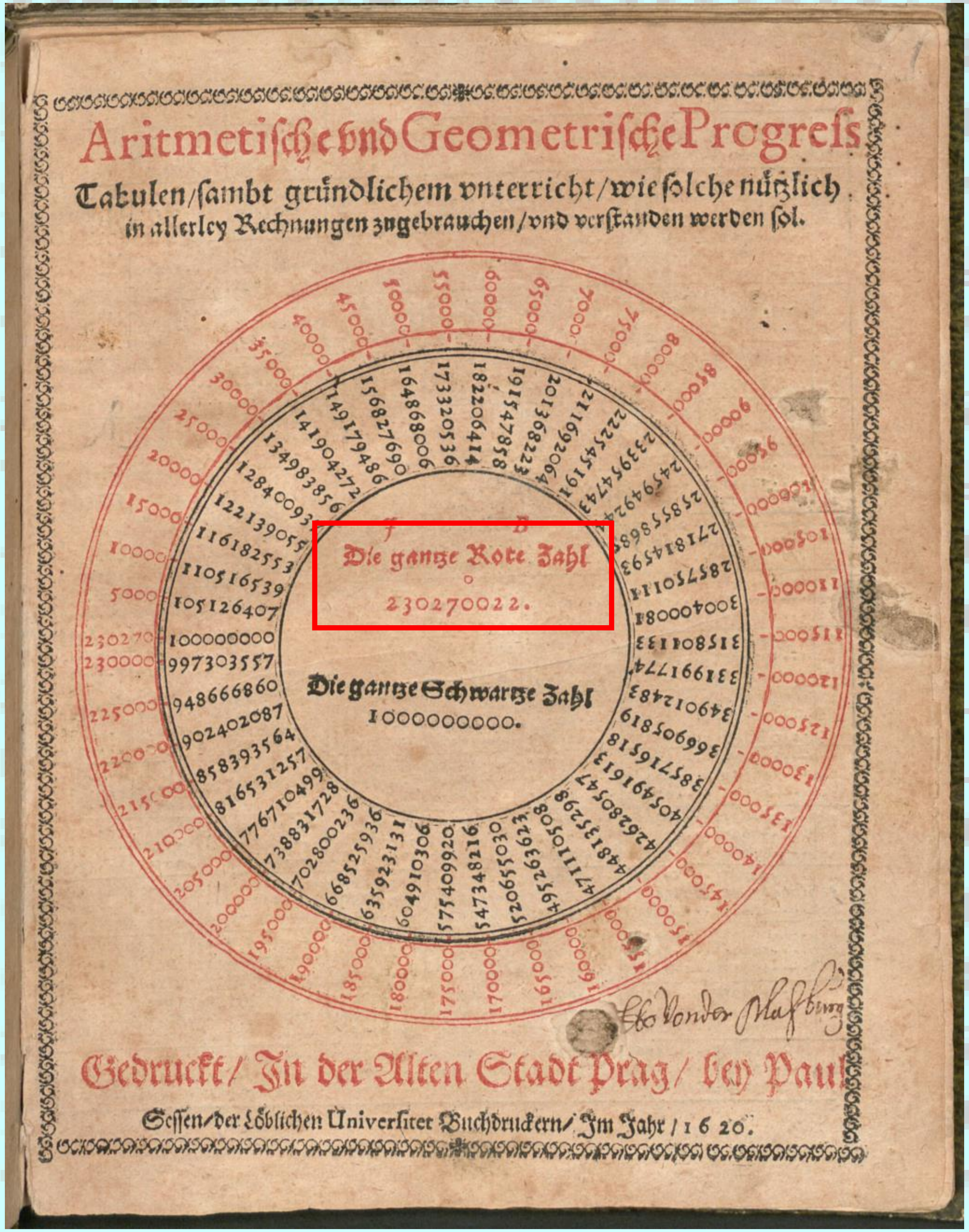
John Napier,
Mirifici logarithmorum canonis constructio, 1619, 32
(table by William Oughtred)

Sinuum proportiones datæ.	Artificialium reſpondentes differentiæ.	Sinuum proportiones datæ.	Artificialium reſpondentes differentiæ.
Dupla	6931469.22	8000pla	89871934.68
Quadrupla	13862938.44	10000pla	92103369,36
Octupla	20794407.66	20000pla	99034838.58
Decupla	23025842.34	40000pla	105966307.80
20cupla	29957311.56	80000pla	112897777.02
40cupla	36888780,78	100000pla	115129211.70
80cupla	43820250.00	200000pla	122060680.92
Centupla	46051684.68	400000pla	128992150.14
200pla	52983153.90	800000pla	135923619.36
400pla	59914623 12	1000000pla	138155054.04
800pla	66846092.34	2000000pla	145086523.26
Millecupla	69077527.02	4000000pla	152017992.48
2000pla	76008996,24	8000000pla	158949461.70
4000 pla	82940465,46	10000000pla	161180896.38

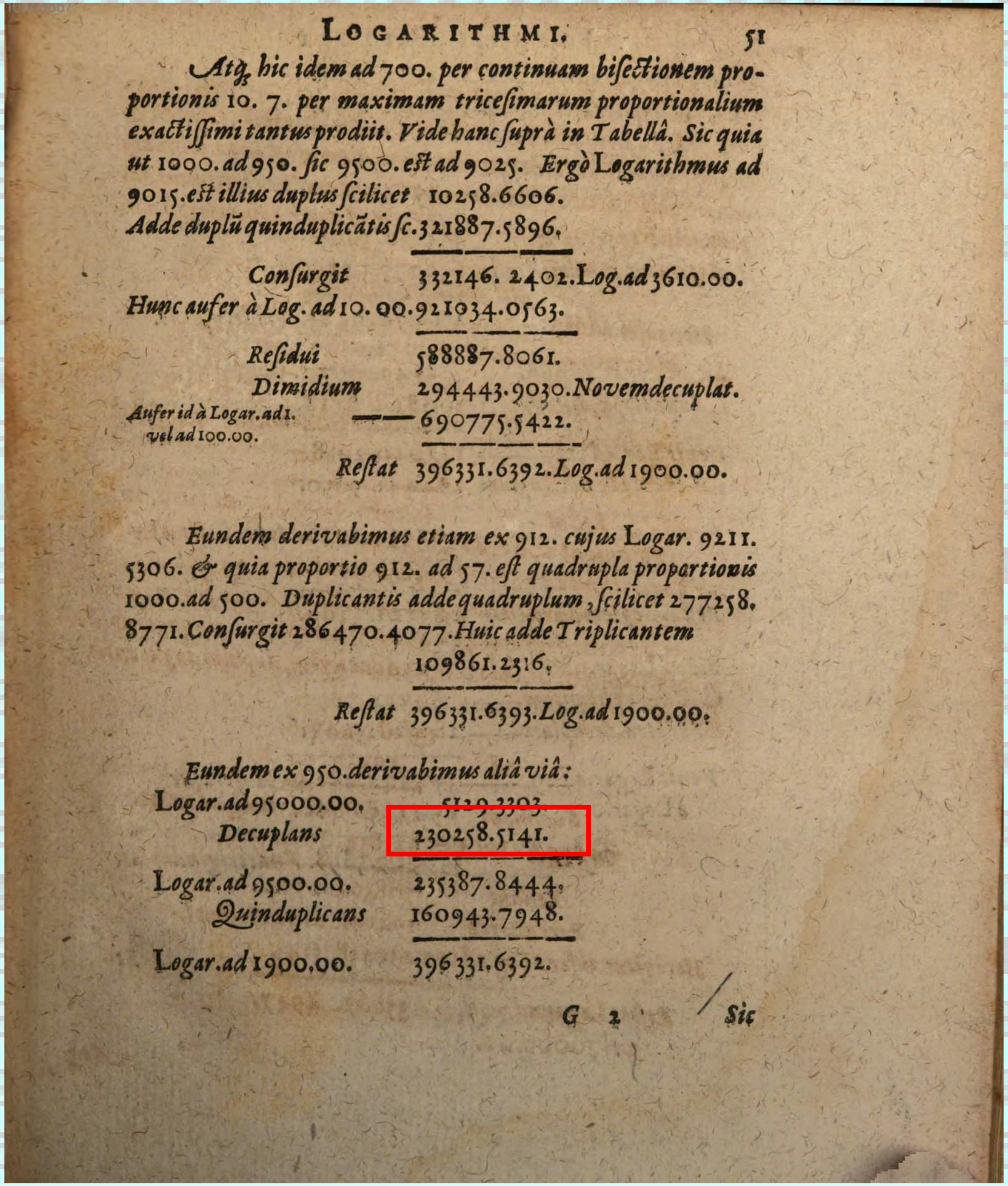
Jost Bürgi, *Aritmetische und Geometrische Progress Tabulen*, 1620

Denis Roegel: Bürgi's "Progress Tabulen" (1620): logarithmic tables without logarithms, 2010 (last updated: 10 January 2013)
https://locomat.loria.fr/buergi1620/buergi1620doc.pdf

$$r(x) = 10\left(\frac{\ln(x/10^8)}{\ln(1.0001)}\right)$$



Johannes Kepler, *Chilias Logarithmorum*, 1624



LOGARITHMI. 51

Atq; hic idem ad 700. *per continuam bisectionem proportionis* 10. 7. *per maximam tricesimarum proportionalium exactissimi tantus prodiit. Vide hanc suprà in Tabellâ. Sic quia ut* 1000. *ad* 950. *sic* 9500. *est ad* 9025. *Ergò* Logarithmus *ad* 9015. *est illius duplus scilicet* 10258.6606.
Adde duplũ quinduplicãtis sc. 321887.5896.

Consurgit	332146. 2402. *Log. ad* 3610.00.
Hunc aufer à Log. ad 10.00.	921034.0563.
Residui	588887.8061.
Dimidium	294443.9030. *Novemdecuplat.*
Aufer id à Logar. ad 1. *vel ad* 100.00.	690775.5422.
Restat	396331.6392. *Log. ad* 1900.00.

Eundem derivabimus etiam ex 912. *cujus* Logar. 9211.5306. *& quia proportio* 912. *ad* 57. *est quadrupla proportionis* 1000. *ad* 500. *Duplicantis adde quadruplum, scilicet* 277258.8771. *Consurgit* 286470.4077. *Huic adde Triplicantem* 109861.2316.

Restat 396331.6393. *Log. ad* 1900.00.

Eundem ex 950. *derivabimus aliâ viâ:*

Logar. ad 95000.00.	5129.3303.
Decuplans	230258.5141.
Logar. ad 9500.00.	235387.8444.
Quinduplicans	160943.7948.
Logar. ad 1900.00.	396331.6392.

G 2 Sic

Henry Briggs,
Arithmetica Logarithmica, 1624, 11

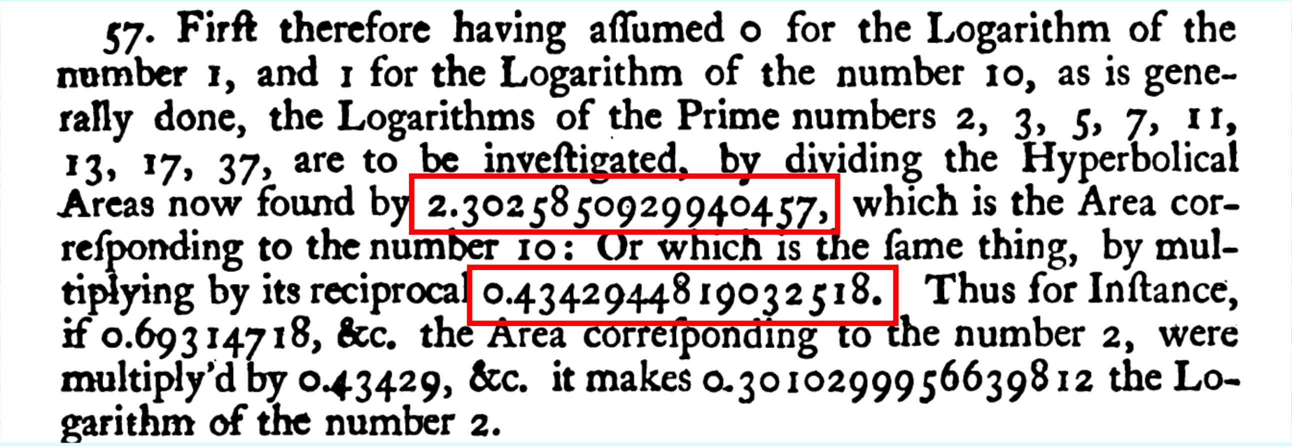
	Numeri continue proportionales supra Vnitatem.	Logarithmi.
	1 Vnitas	0,00000
P	10000,00000,00000,01278,19149,32003,23442	0,00000,00000,00000,05551,11512,31257,82702,12
N	10000,00000,00000,02556,38298,64006,47047	0,00000,00000,00000,11102,23024,62515,65404,24
	10000,00000,00000,03834,57447,96009,70815	0,00000,00000,00000,16653,34536,93773,48106,35
M	10000,00000,00000,05112,76597,28012,94747	0,00000,00000,00000,22204,46[illegible]40,25031,30808,47
	10000,00000,00000,06390,95746,60016,18842	0,00000,00000,00000,27755,57561,56289,13510,59
	10000,00000,00000,07669,14895,92019,43101	0,00000,00000,00000,33306,69703,87546,96212,71
	10000,00000,00000,08947,34045,24022,67523	0,00000,00000,00000,38857,80586,18804,78914,83
L	10000,00000,00000,10225,53194,56025,92108	0,00000,00000,00000,44408,92098,50062,61616,95
X	10000,00000,00000,01	0,00000,00000,00000,04342,94481,90325,1804

C 2

https://books.google.de/books?id=-XR-2oDEb9oC&hl=de

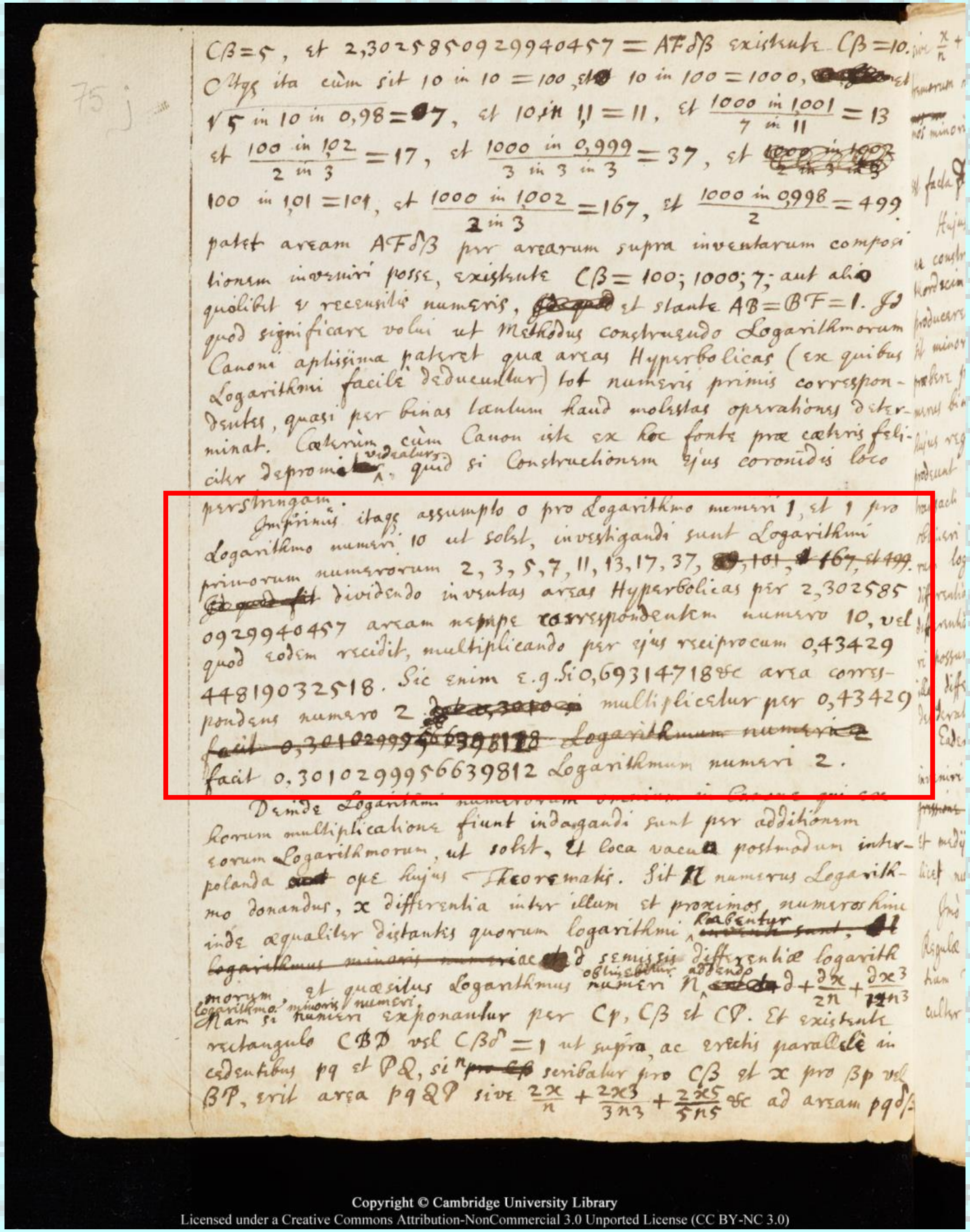
Isaac Newton, *De methodis serierum et fluxionum*, 1670/1

English translation printed in 1736:



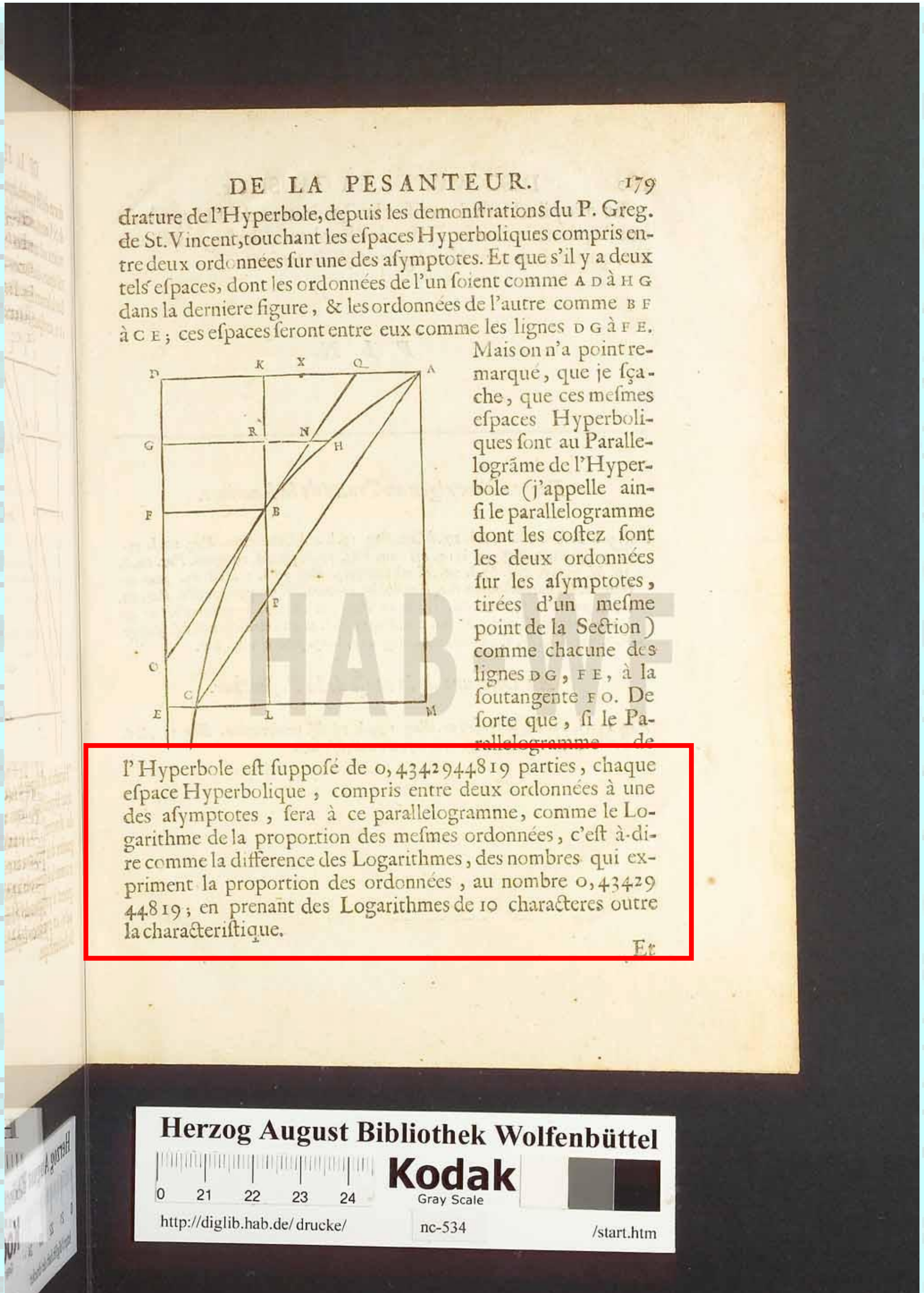
57. First therefore having assumed 0 for the Logarithm of the number 1, and 1 for the Logarithm of the number 10, as is generally done, the Logarithms of the Prime numbers 2, 3, 5, 7, 11, 13, 17, 37, are to be investigated, by dividing the Hyperbolical Areas now found by 2.302585092994045**7**, which is the Area corresponding to the number 10: Or which is the same thing, by multiplying by its reciprocal 0.4342944819032518. Thus for Instance, if 0.69314718, &c. the Area corresponding to the number 2, were multiply'd by 0.43429, &c. it makes 0.3010299956639812 the Logarithm of the number 2.

https://books.google.de/books?id=WyQOAAAAQAAJ&hl=de



Christiaan Huygens, *De la cause de la pesanteur*, 1690

http://diglib.hab.de/drucke/nc-534/start.htm?image=00195



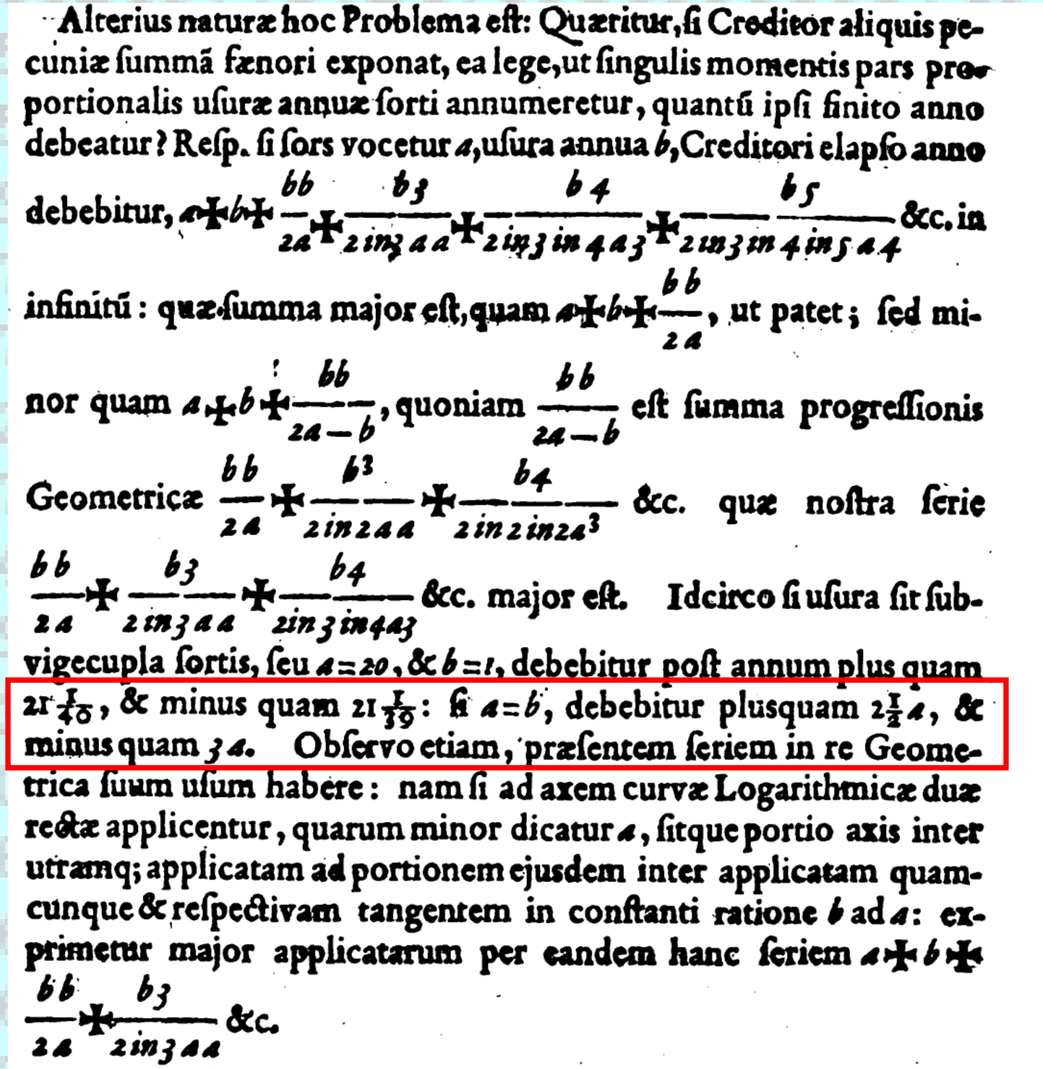
drature de l'Hyperbole, depuis les demonſtrations du P. Greg. de St. Vincent, touchant les eſpaces Hyperboliques compris entre deux ordonnées ſur une des aſymptotes. Et que s'il y a deux tels eſpaces, dont les ordonnées de l'un ſoient comme A D à H G dans la derniere figure, & les ordonnées de l'autre comme B F à C E; ces eſpaces ſeront entre eux comme les lignes D G à F E. Mais on n'a point remarqué, que je ſçache, que ces meſmes eſpaces Hyperboliques ſont au Parallelogrãme de l'Hyperbole (j'appelle ainſi le parallelogramme dont les coſtez ſont les deux ordonnées ſur les aſymptotes, tirées d'un meſme point de la Section) comme chacune des lignes D G, F E, à la ſoutangente F O. De ſorte que, ſi le Parallelogramme de l'Hyperbole eſt ſuppoſé de 0,434294481 9 parties, chaque eſpace Hyperbolique, compris entre deux ordonnées à une des aſymptotes, ſera à ce parallelogramme, comme le Logarithme de la proportion des meſmes ordonnées, c'eſt à-dire comme la difference des Logarithmes, des nombres qui expriment la proportion des ordonnées, au nombre 0,43429 44819; en prenant des Logarithmes de 10 characteres outre la characteriſtique.

Et

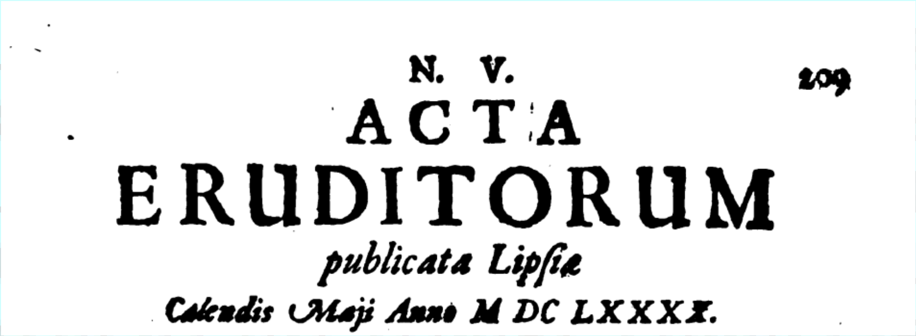
e in approximation

$e = 2.7182818\ldots$

Jacob Bernoulli, *Quaestiones nonnullae de usuris, cum solutione problematis de sorte alearum*, *ACTA ERUDITORUM*, 1690, 219-223



Alterius naturæ hoc Problema eſt: Quæritur, ſi Creditor aliquis pecuniæ ſummã fænori exponat, ea lege, ut ſingulis momentis pars proportionalis uſuræ annuæ ſorti annumeretur, quantũ ipſi finito anno debeatur? Reſp. ſi ſors vocetur a, uſura annua b, Creditori elapſo anno debebitur, $a + b + \frac{bb}{2a} + \frac{b3}{2 \, in \, 3 \, aa} + \frac{b4}{2 \, in \, 3 \, in \, 4 \, a3} + \frac{b5}{2 \, in \, 3 \, in \, 4 \, in \, 5 \, a4}$ &c. in infinitũ: quæ ſumma major eſt, quam $a + b + \frac{bb}{2a}$, ut patet; ſed minor quam $a + b + \frac{bb}{2a - b}$, quoniam $\frac{bb}{2a - b}$ eſt ſumma progreſſionis Geometricæ $\frac{bb}{2a} + \frac{b^3}{2 \, in \, 2 \, aa} + \frac{b4}{2 \, in \, 2 \, in \, 2a^3}$ &c. quæ noſtra ſerie $\frac{bb}{2a} + \frac{b3}{2 \, in \, 3 \, aa} + \frac{b4}{2 in \, 3 \, in \, 4 a3}$ &c. major eſt. Idcirco ſi uſura ſit ſubvigecupla ſortis, ſeu $a = 20$, & $b = 1$, debebitur poſt annum plus quam $21\frac{1}{40}$, & minus quam $21\frac{1}{39}$: ſi $a = b$, debebitur plusquam $2\frac{1}{2}a$, & minus quam $3a$. Obſervo etiam, præſentem ſeriem in re Geometrica ſuum uſum habere: nam ſi ad axem curvæ Logarithmicæ duæ rectæ applicentur, quarum minor dicatur a, ſitque portio axis inter utramq; applicatam ad portionem ejusdem inter applicatam quamcunque & reſpectivam tangentem in conſtanti ratione b ad a: exprimetur major applicatarum per eandem hanc ſeriem $a + b + \frac{bb}{2a} + \frac{b3}{2 \, in \, 3 \, aa}$ &c.



N. V. 209

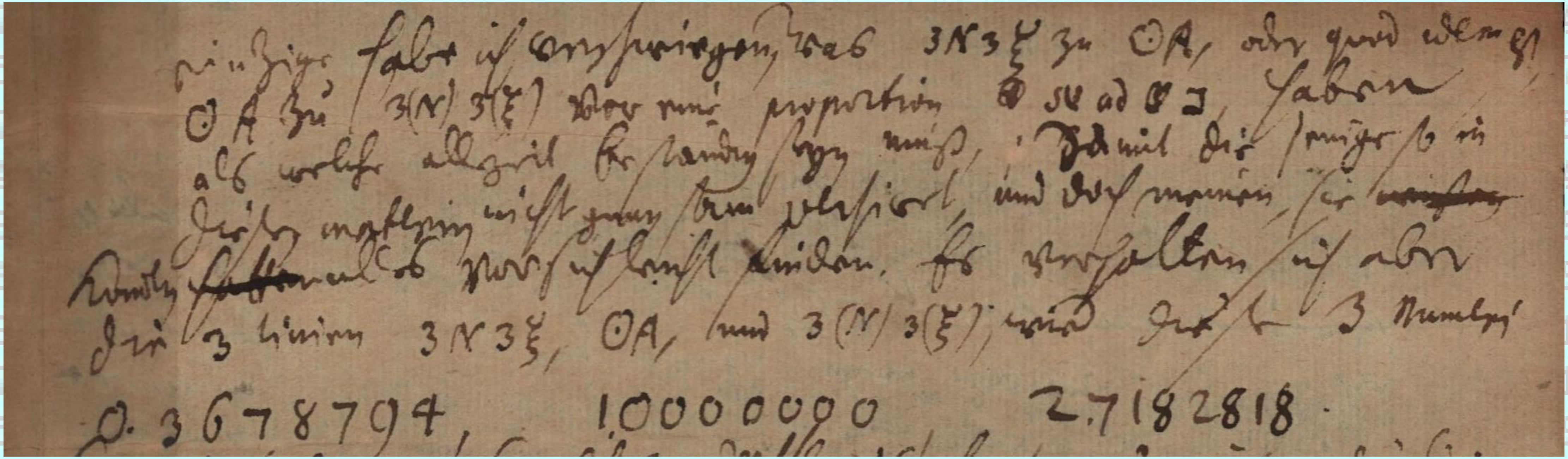
ACTA ERUDITORUM

publicata Lipſiæ

Calendis Maji Anno M DC LXXXX.

https://books.google.de/books?id=nVcg2IyNRsAC&hl=de&redir_esc=y

Gottfried Wilhelm Leibniz to Rudolf Christian von Bodenhausen, 12/22 June 1691


0. 3678794, 1,0000000 2,7182818.

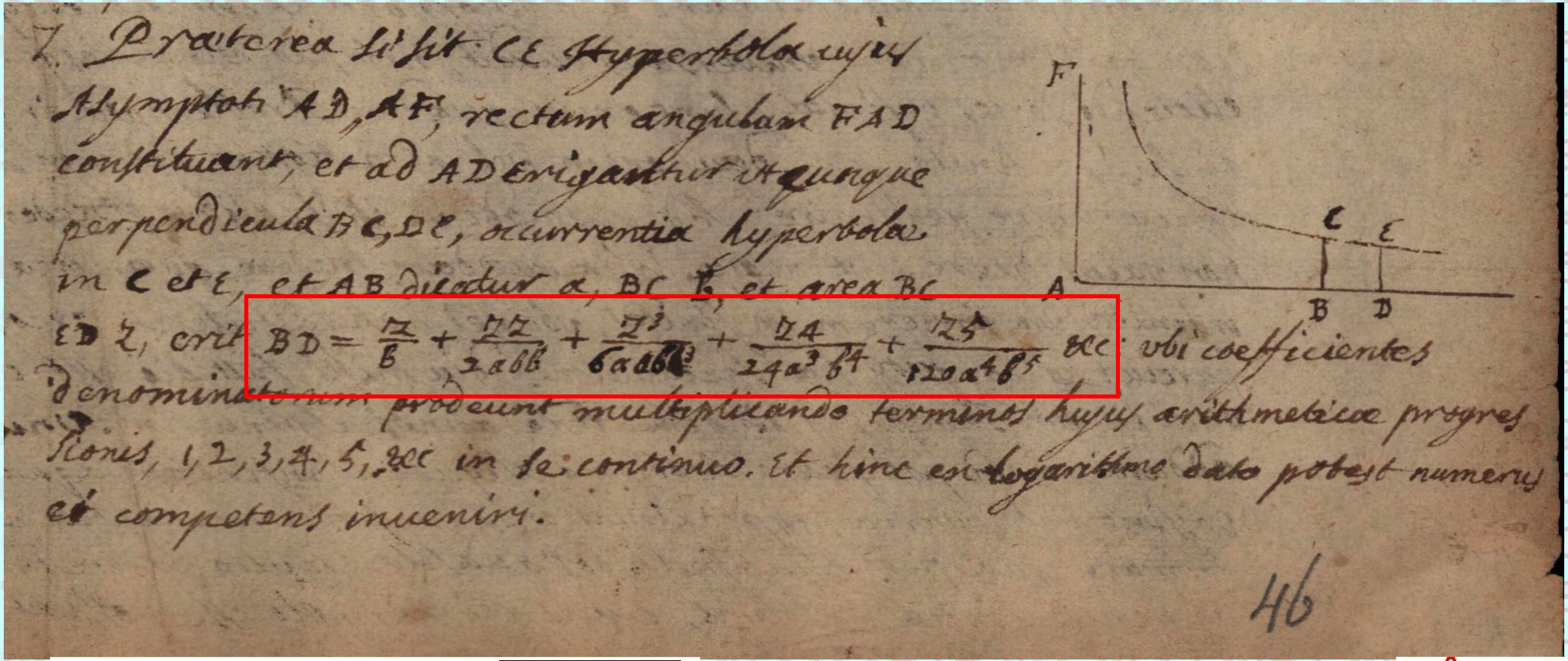
Leibniz writes to von Bodenhausen: “The only thing I withheld [from the publication] is the proportion ℵ to ℶ of $\overline{3N3\xi}$ to $\overline{3(N)3(\xi)}$ . . . The three lines $\overline{3N3\xi}$, $\overline{\theta A}$, $\overline{3(N)3(\xi)}$ are to each other like these three numbers 0.3678794, 1.0000000, 2.7182818.”

Michael Raugh, Siegmund Probst, The Leibniz catenary and approximation of e — an analysis of his unpublished calculations, Historia Mathematica, Volume 49, 2019, 1-19, https://doi.org/10.1016/j.hm.2019.06.001

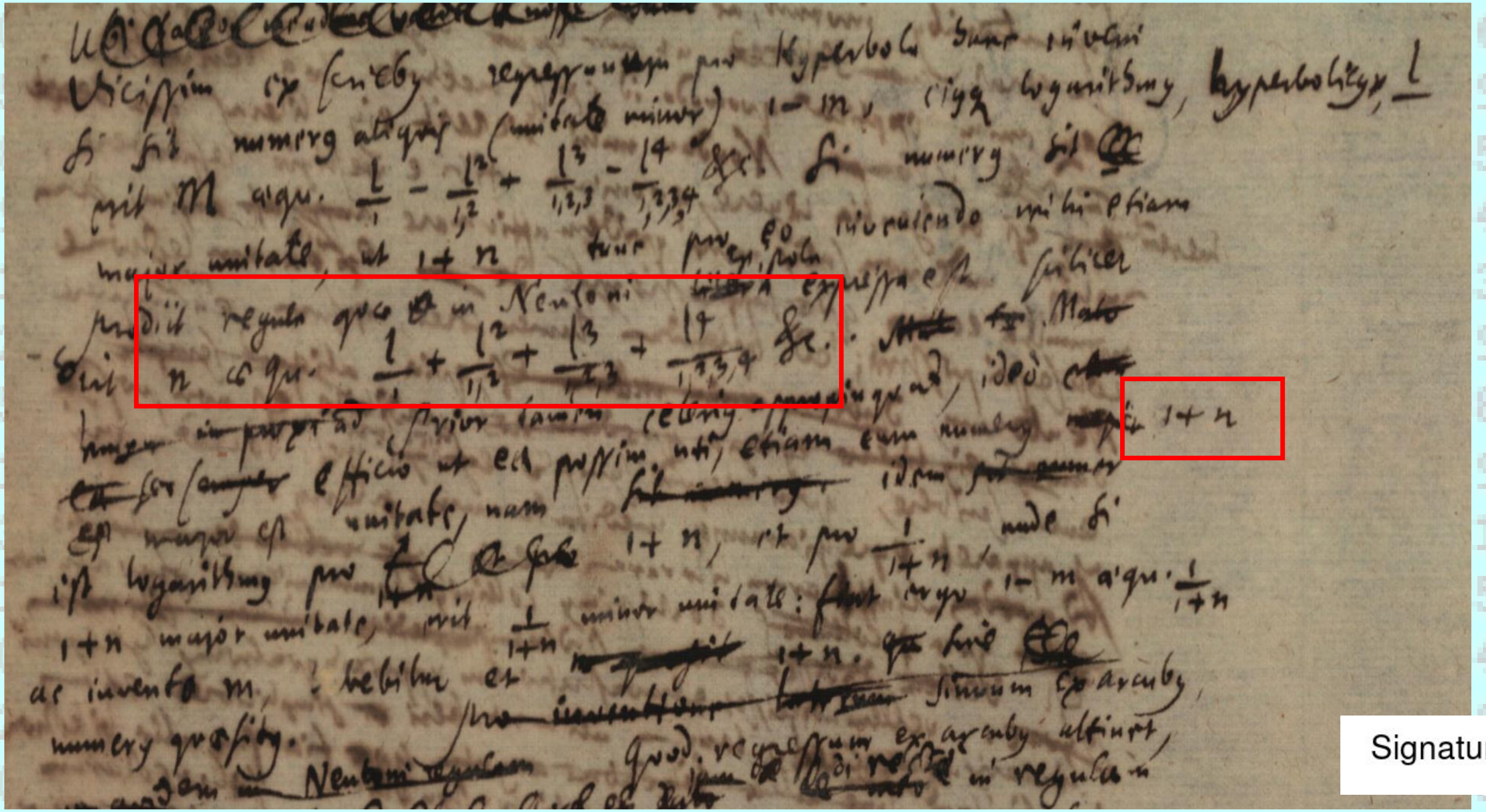
The base of the natural logarithm

$1 + n$ and b come first

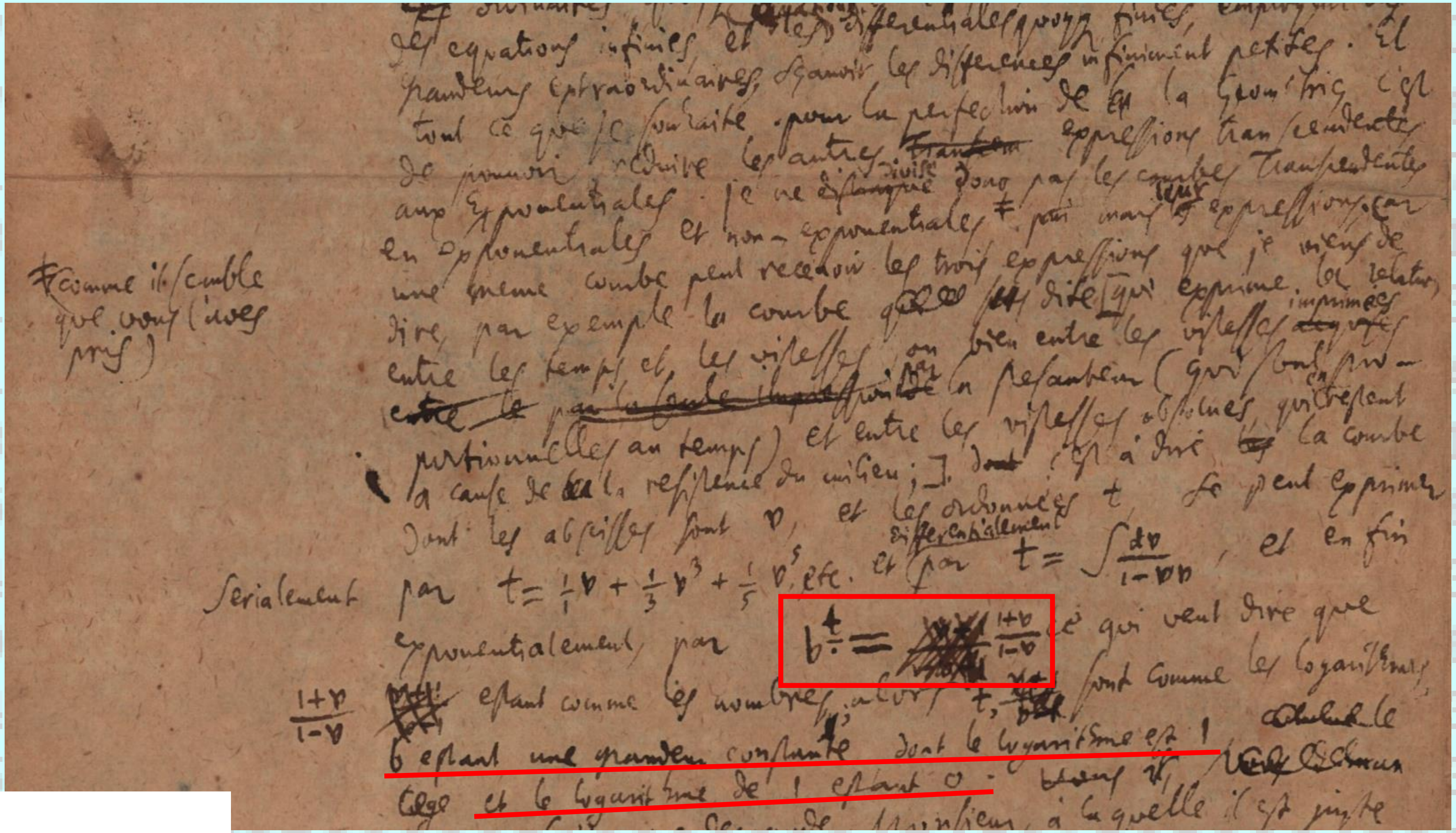
Copy of Isaac Newton’s “Epistola prior” (13/23 June 1676), sent to Gottfried Wilhelm Leibniz by Henry Oldenburg (26 July / 5 August 1676)


7. Praeterea si sit CE Hyperbola cujus Asymptoti AD, AF, rectum angulum FAD constituant, et ad AD erigantur utcunque perpendicula BC, DE, occurrentia hyperbolae in C et E, et AB dicatur a, BC b, et area BCED z, erit $BD = \frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5}$ &c ubi coefficientes denominatorum prodeunt multiplicando terminos hujus arithmeticae progressionis, 1, 2, 3, 4, 5, &c in se continuo. Et hinc ex logarithmo dato potest numerus ei competens inveniri.

Gottfried Wilhelm Leibniz, letter to Henry Oldenburg (draft), August 27, 1676



Gottfried Wilhelm Leibniz to Christiaan Huygens, January 27 / (February 6) 1691 (draft letter)


des equations infinies et des differentiales pour finies, employant les
grandeurs extraordinaires, scavoir les differences infiniment petites. Et
tout ce que je souhaite pour la perfection de la Geometrie c'est
de pouvoir reduire les autres expressions transcendentes
aux exponentiales. Je ne divise donc pas les courbes transcendentes
en exponentiales et non-exponentiales ‡ mais leurs expressions, car
une même courbe peut recevoir les trois expressions que je viens de
dire, par exemple la courbe qui exprime la relation
entre les temps et les vistesses ou bien entre les vistesses imprimées
[illegible] à la pesanteur (qui sont proportionnelles au temps) et entre les vistesses absolues, qui estant
[illegible]
à cause de la resistence du milieu; c'est à dire la courbe
dont les abscisses sont v, et les ordonnées t se peut exprimer differentialement par $t = \int \frac{dv}{1-vv}$, et enfin
Serialement par $t = \frac{1}{1}v + \frac{1}{3}v^3 + \frac{1}{5}v^5$ etc. et
exponentialement par $b^t = \frac{1+v}{1-v}$ ce qui veut dire que
$\frac{1+v}{1-v}$ estant comme les nombres, alors t, sont comme les logarithmes,
b estant une grandeur constante dont le logarithme est 1
et le logarithme de 1 estant 0 [illegible]

‡ comme il semble que vous l'avez pris)

Gottfried Wilhelm Leibniz to Christiaan Huygens, January 27 / (February 6) 1691

And all I wish for the perfection of Geometry is to be able to reduce the other transcendental expressions to Exponentials. I do not therefore divide the Transcendental curves into Exponentials and non-exponentials, (as it seems you have taken it) but their expressions. For one and the same curve can receive the three expressions I have just mentioned, for example the above-mentioned curve […] that is to say, the curve whose abscissae are *v* and whose ordinates are *t* can be expressed serially by $t = \frac{1}{1}v + \frac{1}{3}v^3 + \frac{1}{5}v^5$ etc. And differentially by $t = \int \frac{dv}{1-vv}$, and finally exponentially by $b^{\frac{t}{\cdot}} = \frac{1+v}{1-v}$ [,]* which means that $\frac{1+v}{1-v}$ being like numbers, *t* are like logarithms; *b* being a constant quantity[,] whose logarithm is 1; and the logarithm of 1 being 0.

Translated with www.DeepL.com/Translator (free version)

** Remark by Hans Fischer:* In fact, $b^{\frac{2t}{\cdot}} = \frac{1+v}{1-v}$.

Excerpt from: The Enigmatic Number *e*

Sarah Glaz (University of Connecticut), "The Enigmatic Number [i]e:[/i] A History in Verse and Its Uses in the Mathematics Classroom - The Annotated Poem," *Convergence* (November 2010), DOI:10.4169/loci003482

The Enigmatic Number e
by Sarah Glaz

It ambushed Napier at Gartness,
like a swashbuckling pirate
leaping from the base.
He felt its power, but never realized its nature.
e's first appearance in disguise—a tabular array
of values of *ln*, was logged in an appendix
to Napier's posthumous publication.
Oughtred, inventor of the circular slide rule,
still ignorant of *e*'s true role,
performed the calculations.
A hundred thirteen years the hit and run goes on.
There and not there—elusive *e*,
escape artist and trickster,
weaves in and out of minds and computations:
Saint-Vincent caught a glimpse of it under rectangular
hyperbolas;
Huygens mistook its rising trace for logarithmic curve;
Nicolaus Mercator described its *log* as natural
without accounting for its base;
Jacob Bernoulli, compounding interest continuously,
came close, yet failed to recognize its face;
and Leibniz grasped it hiding in the maze of calculus,
natural basis for comprehending change—but
misidentified as *b*.

Thank you!